Green's Function

1 A Floating Oil Tanker

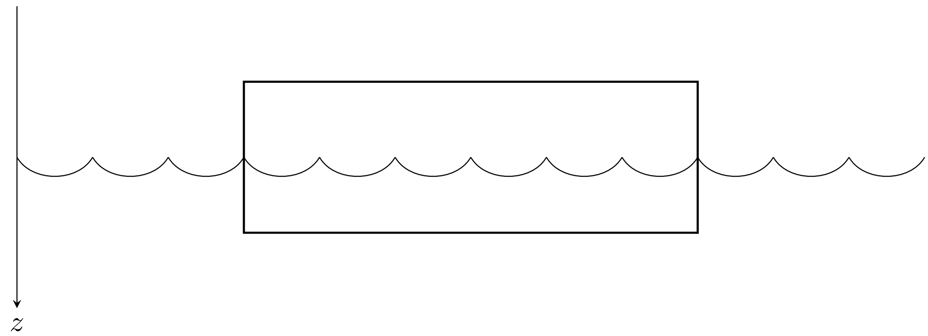
A ship is floating in the middle of the sea and goes up and down, up and down (c.f. sheet 3 of *IA Differential Equations*). In fact, it is undergoing simple harmonic motion, which we can show using Archimedes' Principle.

Archimedes: the upward force is equal to the weight of water displaced.

We model the depth of the ship using depth $z(t)$, where $z = 0$ means the boat is at equilibrium height and $z > 0$ means the ship is *below* the equilibrium. If the sea has density ρ, and the ship has mass M and cross-sectional area A, then the upward force is $g\rho Az$, and hence by Newton's second law,

$$M\ddot{z} = -g\rho Az.$$

The minus sign says the direction of the displacement is always opposite to the force.



This is a second order ODE

$$\ddot{z} + \frac{g\rho A}{M}z = 0$$

and we can solve this:

$$z(t) = \alpha\cos(\omega t) + \beta\sin(\omega t),$$

where $\omega = \sqrt{\frac{g\rho A}{M}}$ is the angular frequency (if $\omega = 2\pi$ then one oscillation is completed in 1 second).

The coefficients α and β are determined uniquely by the initial conditions. For example, if the boat was at depth 1 initially with no initial velocity, then

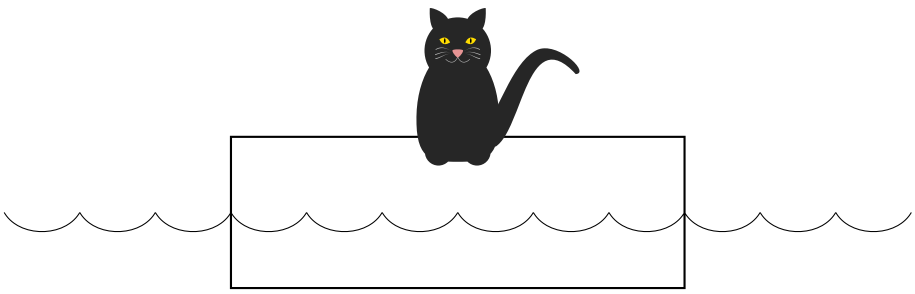
$$z(0) = 1,\ \dot{z}(0) = 0 \implies z(t) = \cos(\omega t).$$

So a nice, comfortable cosine. For the passengers who feel nauseous even at the slightest perturbation, we had better start the ship at equilibrium with no initial velocity. Then

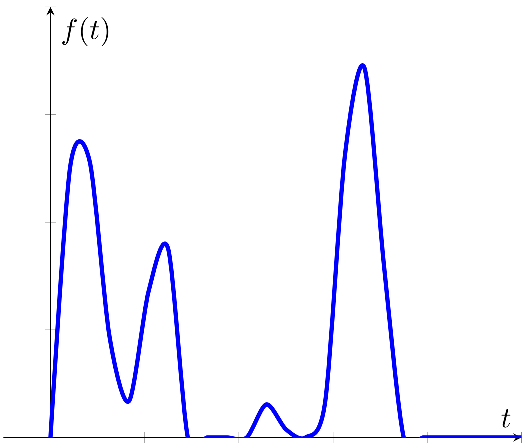
$$z(0) = 0,\ \dot{z}(0) = 0 \implies z(t) = 0.$$

An unperturbed ship remains perfectly still in still water.

But then a domestic cat appears and begins to perturb it!



It's Schrödinger's cat. He was not happy being trapped inside a box for the first experimental run, and decided to come out to the deck of the oil tanker to let off some steam before he has to go back in for a second round of experiment. Being very unpredictable, he applies a wildly erratic force $f(t)$ to the ship:



Oh dear. Looks like he is throwing a tantrum. The question is: what is the depth of the ship now? We have an equation for that. Back to Newton's second law the force on the ship is a combination of the buoyancy force and the cat's, so:

$$M\ddot{z} = -g\rho A z + f(t).$$

Rearranging gives

$$\ddot{z} + \omega^2 z = \frac{1}{M} f(t). \tag{$*$}$$

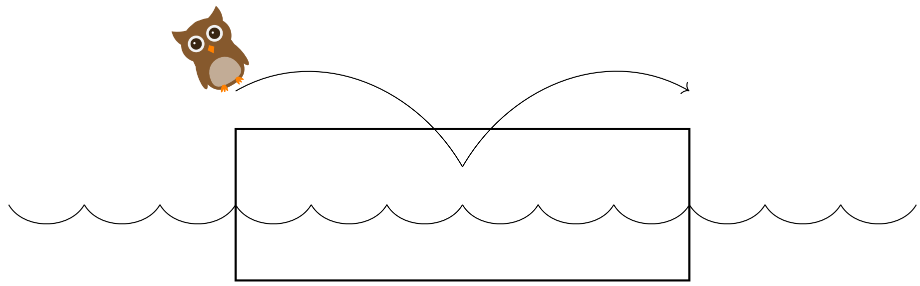
Thus we have a *forced* wave equation to solve. In *IA Differential Equations*, we solve this by finding the complementary function: no problem, it is a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ as before, and then *guessing* a particular integral $z_p(t)$. However, the cat is producing a very strange force which cannot be written easily as elementary functions, so we have no idea how to get this particular integral. How should we proceed?

2 Green's Brilliant Idea

The captain runs across the ship to George Green's cabin to inform him of the unfortunate situation. Green looks at the equation and tells the captain not to be afraid. He scribbles out equation ($*$) on a large piece of brown paper with a black marker (as he does on his youtube channel) and stares at it for a while. He then crosses off the forcing term and replaces it with a Dirac delta:

$$\ddot{z} + \omega^2 z = \delta(t - \tau). \tag{OWL}$$

The Dirac delta models an impulse at τ.



This models the effect of a bird landing on the ship at time τ and then taking off.

Captain: What good is that? We don't care about birds; we have a *cat*astrophe at the moment!

Mr Green ignores him, and continues to solve the equation involving a bird landing. There is no disturbance apart from $t = \tau$, so the ship undergoes harmonic motion before and after τ:

$$z(t) = \begin{cases} \alpha_1 \cos(\omega t) + \beta_1 \sin(\omega t) & t < \tau \\ \alpha_2 \cos(\omega t) + \beta_2 \sin(\omega t) & t > \tau. \end{cases}$$

The coefficients $\alpha_1, \beta_1, \alpha_2, \beta_2$ depend on τ and we have to find them. But the boat was stationary at $t = 0$, so in fact it will be stationary for all times up to τ:

$$z(t) = \begin{cases} 0 & t < \tau \\ \alpha_2 \cos(\omega t) + \beta_2 \sin(\omega t) & t > \tau. \end{cases}$$

The impulse at time τ initiates some initial velocity of the ship just after time τ. By integrating equation (OWL) between time τ^- and τ^+, we get

$$\dot{z}(\tau^+) - \dot{z}(\tau^-) = 1.$$

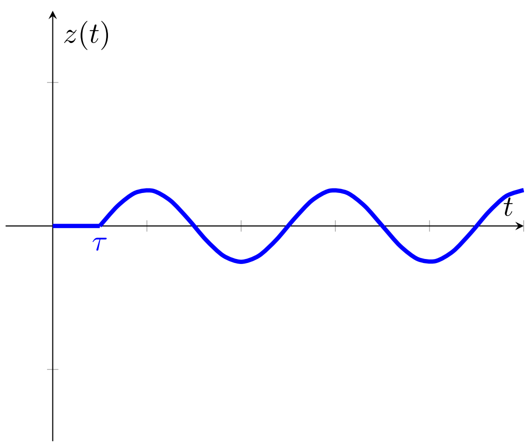
Since $\dot{z}(\tau^-) = 0$ (boat was not moving before time τ), we have $\dot{z}(\tau^+) = 1$. By continuity, we have $z(\tau^+) = 0$ (the ship is still at height 0 just after the impulse, as ships don't teleport). This gives us two initial conditions to find α_2 and β_2. However, he is clever and decides to rewrite the linear combination as

$$z(t) = \begin{cases} 0 & t < \tau \\ \alpha_3 \cos(\omega(t-\tau)) + \beta_3 \sin(\omega(t-\tau)) & t > \tau. \end{cases}$$

This is still the most general form, except we have traded in α_2, β_2 for α_3, β_3. Now $z(\tau^+) = 0 \implies \alpha_3 = 0$, and $\dot{z}(\tau^+) = 1 \implies \beta_3 = 1/\omega$, so

$$z(t) = \frac{1}{\omega} \sin(\omega(t-\tau)) 1_{t > \tau}.$$

The graph looks like



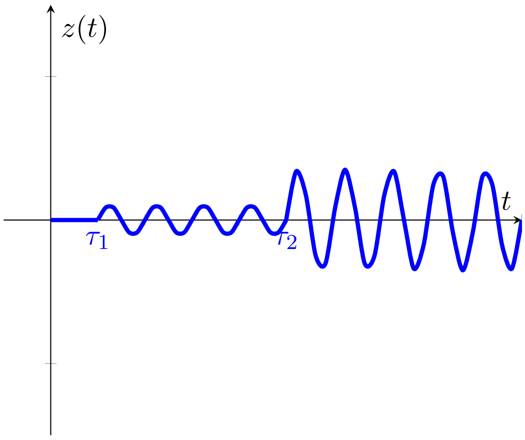
We see that at time τ the boat sinks (remember $z > 0$ means the ship goes downwards, because z is depth), and then oscillates with simple harmonic motion after that.

Now we move onto a slightly more advanced situation. What if we have two owls, each attacking the ship at different times and different intensities? Then we are to solve

$$\ddot{z} + \omega^2 z = f_1 \delta(t - \tau_1) + f_2 \delta(t - \tau_2),$$

where τ_i is the attacking time of the ith owl and the constants f_i represent the intensity. Fortunately, we do not need to do any more calculations. We know how to solve the solution for one owl, so if we have two owls we can simply add the two solutions together!

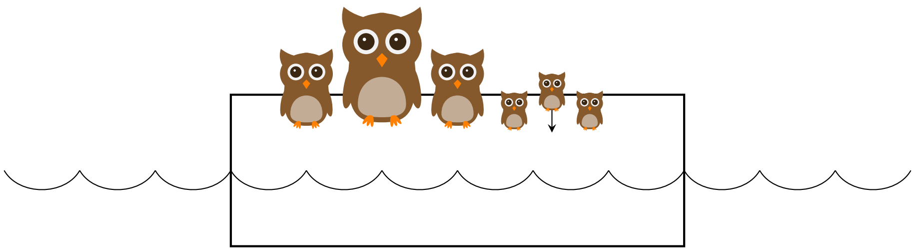
$$z(t) = \frac{1}{\omega} \left[f_1 \sin(\omega(t - \tau_1)) 1_{t > \tau_1} + f_2 \sin(\omega(t - \tau_2)) 1_{t > \tau_2} \right]$$



We can see there are two impulses. In the plot above, the second one is more intense than the first one.

What about finitely many owls? No problem:

$$z(t) = \frac{1}{\omega} \sum_{i=1}^{n} f_i \sin(\omega(t - \tau_i)) 1_{t > \tau_i}.$$



The ship suddenly jerks violently, knocking over a candle which sets the brown paper aflame. Green is so deep in thought that he doesn't even notice. It is only when the paper has burnt completely into ash that he suddenly stands up and declares:

A cat is simply a superposition of uncountably many owls!

Captain: What do you mean?

Green explains while he finds his balance. A general forcing term $f(t)$ can be decomposed into a superposition of Dirac deltas:

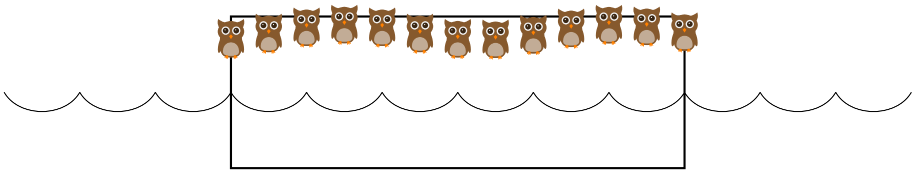
$$f(t) = \int_0^\infty f(\tau) \delta(t - \tau) \, d\tau.$$

This equation is a result of the sampling property of δ, but we can interpret it in a different way. We can think of the coefficient $f(\tau)$ as the intensity of the τth owl, who is applying a force $\delta(t-\tau)$ to the ship. By adding up the solution for the τth owl, which is

$$f(\tau)\frac{1}{\omega}\sin(\omega(t-\tau))1_{t>\tau},$$

the overall solution must be

$$z(t)=\int_0^\infty f(\tau)\frac{1}{\omega}\sin(\omega(t-\tau))1_{t>\tau}\,d\tau\,!$$



This is my attempt to draw uncountably many owls. However, the LaTeXdocument didn't compile so I'll have to make do with this one.

3 Initial Value Problems

While the Captain hurries off to prevent the sinking of his ship, let's try to generalise Green's idea. It works for all kinds of differential operators. The idea of superposition will work as long as the differential operator is linear. Let's look at the general formulation. We would like to solve

$$\begin{cases}\mathcal{L}u(t)=f(t) & t\geq 0\\ u(0)=\dot{u}(0)=0. & \end{cases} \tag{IVP}$$

Here $\mathcal{L}$ is a second-order differential operator:

$$\mathcal{L}u(t)=\alpha(t)\frac{d^2u}{dt^2}+\beta(t)\frac{du}{dt}+\gamma(t)u(t).$$

We define *Green's function* $G(t;\tau)$ to be the solution to

$$\begin{cases}\mathcal{L}G(t;\tau)=\delta(t-\tau) & t\geq 0\\ G(0;\tau)=\dot{G}(0;\tau)=0. & \end{cases}$$

Here we have used a semicolon to separate t from τ, as is traditional. We view τ as a parameter rather than another variable, so it may be more helpful conceptually to think of $G(t;\tau)$ as $G_\tau(t)$. What we are doing here is solving for a *family* of solutions with parameter τ. In the case of the wave equation, $G(t;\tau)$ represents the depth of the ship due to an impulse at time τ.

We solve this problem using methods we learnt from *IA Differential Equations*. First we have to find complementary functions. There are two of them, say u_1 and u_2, because $\mathcal{L}$ is second-order. Then since there is no forcing apart from $t=\tau$, we are solving the unforced equation on the intervals $[0,\tau)$ and (τ,∞), giving:

$$G(t;\tau)=\begin{cases}Au_1(t)+Bu_2(t) & t\in[0,\tau)\\ Cu_1(t)+Du_2(t) & t\in(\tau,\infty).\end{cases}$$

But just like the harmonic oscillator example above, the initial condition $G(0;\tau)=\dot{G}(0;\tau)=0$ forces $A=B=0$. Furthermore, it is more advantageous to rewrite $Cu_1(t)+Du_2(t)$ in a cleverer form as Green did:

$$G(t;\tau)=\begin{cases}0 & t\in[0,\tau)\\ Eu_1(t-\tau)+Fu_2(t-\tau) & t\in(\tau,\infty).\end{cases}$$

Thus the Green's function for IVP is always of the form

$$G(t;\tau) = 1_{t>\tau} \cdot \text{(some function)}.$$

Now that we have $G(t;\tau)$, we can simply *write down* the solution for (IVP) by inspection using Green's Brilliant Idea:

$$u(t) = \int_0^\infty f(\tau) G(t;\tau)\, d\tau. \tag{GBI}$$

We can verify this is true:

$$\begin{aligned}
\mathcal{L}u(t) &= \int_0^\infty f(\tau) \mathcal{L} G(t;\tau)\, d\tau && \text{diff. under integral sign} \\
&= \int_0^\infty f(\tau) \delta(t-\tau)\, d\tau && \text{definition of } G(t,\tau) \\
&= f(t) && \text{sampling property.}
\end{aligned}$$

Remark 3.1. A final remark is on *causality*. Note that (GBI) can be written as

$$\begin{aligned}
u(t) &= \int_0^\infty f(\tau) 1_{t>\tau} G(t;\tau)\, d\tau \\
&= \int_0^t f(\tau) G(t;\tau)\, d\tau.
\end{aligned}$$

What I've done here is to notice that $1_{t>\tau}^2 = 1_{t>\tau}$ so that G, which equals to $1_{t>\tau} \cdot$(some function), is unaffected. Then by the property of the indicator function, we can change the integration limit to $[0,t)$. What this says now is that the value of $u(t)$ is determined only by owl-jumps that occurred at before time t, since τ is always less than t in the integral. This is consistent with the fact that owls in the future cannot affect the present.

 $= \int_0^\infty$  $(\tau)\, d\tau$

THE END