Contour Integration

Step 1. Define $f(z)$ you wish to integrate and call the integral I. Always swap sin and cos for an e^{iz} so that you can apply Jordan's Lemma, e.g.

$$I = \int_{-\infty}^{\infty} \frac{\sin z}{z}\,dz = \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz$$

If your function f has a branch cut, define it here, e.g. take $\log z$ to have argument $(-\pi, \pi)$.

Step 2. Draw out your contour, labelling the contours systematically (e.g. $C_0, C_1, \dots$), put in the residues with an x, and draw any necessary branch cuts. Use shorthand, e.g.

$$\int_{C_0} \equiv \int_{C_0} f(z)\,dz.$$

Step 3. Evaluate the integral along every contour systematically.

- Jordan's Lemma: if $|g| \to 0$ as $|z| \to \infty$ for z in the upper half plane, and $a > 0$, then

$$\int_{C_R} e^{iaz} g(z)\,dz \to 0,$$

  where C_R is the semicircular contour $C_R = \{Re^{i\theta} : \theta \in [0, \pi]\}$.
- Indentation Lemma: if z_0 is a simple pole of f and C_ε is a anticlockwise arc of angle α around z_0, then as $\varepsilon \to 0$,

$$\int_{C_\varepsilon} f(z)\,dz \to i\alpha \operatorname*{res}_{z=z_0} f(z).$$

- Use O notation for less clutter.
- For more complicated curves, write out the parametrisation for the curve to prevent careless mistakes.

Step 4. State the nature of each pole and compute every residue inside the curve. Put in the factor of $2\pi i$ here already.

- Simple poles:

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z)$$

$$\operatorname*{res}_{z=z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)}{g'(z)} \qquad \text{very useful}$$

- Pole of order n:

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right]\Big|_{z=z_0}$$

- L'Hopital's rule may be helpful.

Step 5. Put everything together using the residue theorem.

$$\sum_i \int_{C_i} = \pm 2\pi i \sum \text{residues},$$

(negative sign for clockwise contours).