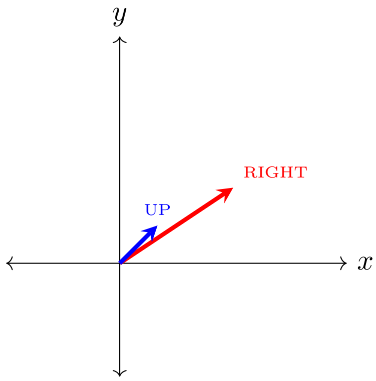
Span, Linear Independence and Basis

1 Span

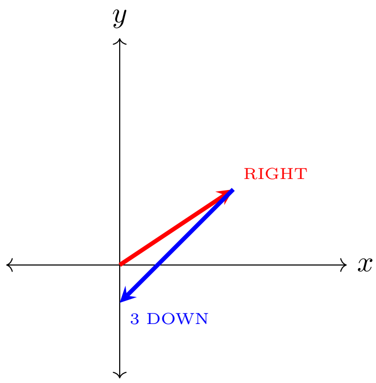
Suppose you live in $\mathbb{R}^2$ and you are sitting in your spaceship at the origin. Let's say you want to get to another point (x, y) in $\mathbb{R}^2$. To navigate around space, you use the arrow on your keyboard. However it turns out the navigation system isn't that easy to use. If you press LEFT or RIGHT, instead of moving horizontally, you move in the $(3, 2)$ direction. If you press UP or DOWN you move along $(1, 1)$. The diagram below represents the spaceship's wonky controls schematically. Question: can you reach your destination (x, y)?



Suppose you want to reach $(0, -1)$. Then you should do the following:

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This means: press RIGHT for 1 second which will take you to $(3, 2)$, then backtrack by pressing DOWN for 3 seconds to get to $(0, -1)$.



In fact you can reach any location on the xy-plane! To reach (a, b), we need to find λ and μ such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Here, a, b are known quantities and λ, μ are unknown. To find λ, μ, we can solve the simultaneous equation

$$\begin{cases} a &= 3\lambda + \mu \\ b &= 2\lambda + \mu. \end{cases}$$

Subtract (1) $-$ (2) to get $a - b = \lambda$, and substitute into equation (2) to get $\mu = b - 2\lambda = b - 2(a - b) = -2a + 3b$. So

$$\lambda = a - b$$
$$\mu = -2a + 3b$$

gives you the required set of instructions to navigate to the point (a, b). In the first example, we had taken $a = 0, b = -1$, which indeed gives $\lambda = 1$ and $\mu = -3$.

This captures the definition of span. The span of $(3, 2)$ and $(1, 1)$ is the set of points you can reach using your spaceship, and it is equal to the xy-plane:

$$\operatorname{span}\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \left\{\lambda\begin{bmatrix}3\\2\end{bmatrix} + \mu\begin{bmatrix}1\\1\end{bmatrix} : \lambda, \mu \in \mathbb{R}\right\} = xy\text{-plane}.$$

This is the set of all points you can reach by controlling λ for joystick 1 and μ for joystick 2 (whether forwards or backwards).

Now what if we equipped our spaceship with more engines which allows us to move in more directions?

Definition 1.1 (Span)**.** Let $v_1, ..., v_n$ be vectors in $\mathbb{R}^d$. The **span** of these vectors is the set

$$\operatorname{span}\{v_1, ..., v_n\} = \{\lambda_1 v_1 + ... + \lambda_n v_n : \lambda_1, ..., \lambda_n \in \mathbb{R}\}.$$

Informally, you can interpret this as:

if your spaceship is equipped with engines with directions $v_1, ..., v_n$, then the span is defined to be the set of all points your spaceship can reach.

Remark 1.2. Note that the span will always contain the origin because that's where you start ($\lambda = 0, \mu = 0$). Also note that span is always a vector subspace (a line, a plane etc.) because you can only move in straight lines. So span will never be a circle for example.

Remark 1.3. Note that the λ's can be negative. This means we should activate the joystick for $|\lambda|$ seconds but in the *reverse direction.*

Example 1.4. Here's a less trivial example. Suppose you're now in $\mathbb{R}^3$, and your spaceship is equipped with engines that let you move in the directions

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

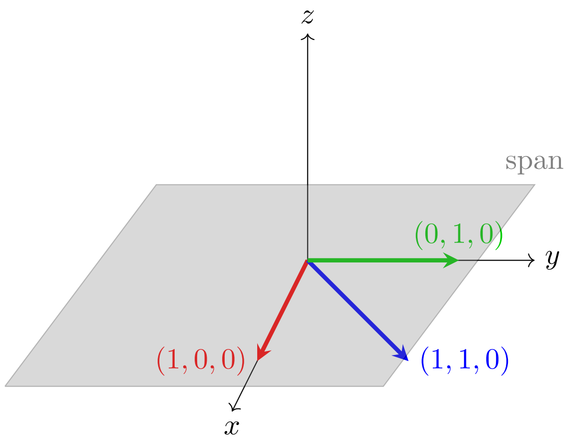
What is the set of points you can reach, i.e. what is the span of these two vectors? Answer: it is the unique plane in 3D that passes through the points $(1, 0, 0)$, $(1, 1, 1)$ and $(0, 0, 0)$ (diagram too difficult to draw so I will omit it).

2 Linear Dependence

Let's say you're in $\mathbb{R}^3$, and your spaceship is equipped with engines that let you move in the directions

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

So you have 3 joysticks that will move you in those directions respectively. Let's first compute the span. What are the points you can reach?



Notice that none of the three engines will allow you to move in the z-direction. Also, it is clear that you can move to any point that lies in the xy plane by using the first and second vector. Thus

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\} = xy\text{-plane}.$$

We have 3 engines, but we actually only ended up with 2 degrees of freedom! We can only move in a 2 dimensional subspace. Why? You realise that the third joystick is useless because you could have reached $(1, 1, 0)$ with just the other two anyway. Indeed, the formula

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

tells you that the third joystick is useless. What the above equation means is that the third joystick is just a "shortcut key" to the instruction "1 second for joystick 1 and 1 second for joystick 2", but it does not actually unlock any new directions for your spaceship.

However, you could equally say that the first joystick is useless, because you can reach $(1, 0, 0)$ using just the second and third vector. There is no reason to discriminate the third one to say it is the only useless one. In order to not single out the third joystick, we move all the vectors on one side of the equation:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This says there is a way to activate each joystick exactly once such that we end back up at the origin. In mathematics, this says that these vectors are **linearly dependent**. Here is the definition:

Definition 2.1 (linear dependence)**.** The vectors $v_1, ..., v_n \in V$ are **linearly dependent** if there exists $\lambda_1, ..., \lambda_n \in \mathbb{R}$, not all zero, such that

$$\lambda_1 v_1 + ... + \lambda_n v_n = 0.$$

In the above case, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$ shows that the three vectors are linearly dependent, i.e., (at least) one of the three is redundant.

Hence the algorithm for finding whether there are redundant joysticks on your spaceship is as follows: attempt to find a way to activate joystick 1 for λ_1 seconds, joystick 2 for λ_2 seconds,..., joystick n for λ_n seconds so that we end back up at the origin. Trivially, $\lambda_1 = \cdots = \lambda_n = 0$ works (not touching the spaceship controls does not move the spaceship so we are still at the origin obviously!). But if there is a nonzero way of doing so, we have "detected redundancy".

Confusio[1]: but for the controls

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we can just go east for one second, and then with the same joystick, go west for one second again? Or even

east 1 second, north 1 second, west 1 second, south 1 second?

Then we have come back to the origin in a non-trivial way?

Professor: We said to only activate each stick once! Doing λ for joystick 1 and then $-\lambda$ for it again doesn't count! Your little east-west journey says

$$1\mathbf{v}_1 - 1\mathbf{v}_1 = 0,$$

and your square loop journey says

$$1\mathbf{v}_1 + 1\mathbf{v}_2 - 1\mathbf{v}_1 - 1\mathbf{v}_2 = 0,$$

and this doesn't prove anything, as it just says $0 = 0$!

3 Linear Independence

The opposite of linear dependence is **linear independence**. The definition is the negation of the above definition:

Definition 3.1 (linear independence)**.** The vectors $v_1, ..., v_n \in V$ are **linearly independent** if whenever $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that

$$\lambda_1 v_1 + ... + \lambda_n v_n = 0,$$

then $\lambda_1 = ... = \lambda_n = 0$.

Here is the algorithm for determining whether your engines all work "independently" and none of them are redundant: suppose there is a way to go back to the origin by activating the joysticks each by λ_i seconds (*exactly once!*). If the only way to do this is to not have done anything at all, then we have shown that none of the engines on our ship is redundant.

[1] This is a character from the series of books by A. Zee. He represents the set of all students who are confused by poor mathematical explanations at university.

4 Computing span

Example 4.1. Suppose your spaceship has 3 engines to move along

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ -12 \end{bmatrix}.$$

Question: what dimension is your span? Are we confined to a plane forever (☹) or can we reach the whole space (☺)? It is not clear from first glance, but we can do something to simplify the problem. Here is the key observation: note that your span won't change if you change the middle vector to

$$\mathbf{v}_{2'} = \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

What we are doing is to replace joystick 2 with a brand new joystick 2' that has the combined instruction of

joystick 2 for 1 second and joystick 1 (in reverse) for 1 second.

Why does this not affect the span, i.e. the places you can reach? Well, you might really miss your original joystick 2, but have no fear! You can recover the "lost" control by doing

joystick 1 for 1 second, joystick 2' for 1 second,

or in vector speak,

$$\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_{2'}.$$

Thus your spaceship was just as good as before: no upgrade, no downgrade. Our spaceship now has engines

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_{2'} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ -12 \end{bmatrix}.$$

Note that it is now a teensy bit easier to calculate the span, because of the appearance of the zero! It is clear to the pilots on board that joystick 2' is not a useful way to go in the x direction. We should welcome our new equipment with vigour and not ostracise it by labelling it with primes. Let's just assimilate it by renaming it $\mathbf{v}_2$:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ -12 \end{bmatrix}.$$

(Note that doing such a renaming thing in mathematics is generally frowned upon as it could lead to confusion. But then the downside to keeping everyone's names constant is that you'll end up with beasts like $\mathbf{v}_{2''''''}$.)

We can do the same thing with the third vector: replace it by

$$\mathbf{v}_{3'} = \mathbf{v}_3 - 3\mathbf{v}_1 = \begin{bmatrix} 6 \\ 3 \\ -12 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -9 \end{bmatrix}.$$

We have bought in a new piece of equipment $\mathbf{v}_{3'}$ that allows us to do joystick 3' for 1 second = joystick 3 for 1 second, joystick 1 (in reverse) for 3 seconds.

Again, no harm throwing away $\mathbf{v}_3$ in favour of $\mathbf{v}_{3'}$, because $\mathbf{v}_3$ can be easily reverse-engineered:

$$\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_{3'}.$$

And there is no upgrade to the ship, as the span remained the same:

$$\operatorname{span}\left\{\begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\2\end{bmatrix}, \begin{bmatrix}6\\3\\-12\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}0\\-3\\-9\end{bmatrix}\right\}.$$

The advantage of the new equipment, however, is that we can now see where we are going more clearly! In order to advance in the x-direction, the only method is to steer with $\mathbf{v}_1$. Touching $\mathbf{v}_2$ or $\mathbf{v}_3$ won't change our x-coordinate because of the zero in the first entry.

There is a bonus observation: we notice obviously the second and third vector do the same thing. They both allow us to steer in the $(0, 1, 3)$ direction. So one of them is redundant, and we can just throw it away. Let's throw away $\mathbf{v}_3$ (goodbye!). Now

$$\operatorname{span}\left\{\begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}0\\-3\\-9\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}2\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right\}.$$

The two vectors we are left with are clearly linearly independent, because $(2, 2, -1)$ lets us move in the x direction and $(0, 1, 3)$ doesn't. So we can only move in a two dimensional vector subspace (☹).

The algorithm can be summarised more succinctly with what's known as "column operations". The steps above can be written as follows in matrix notation:

$$\begin{bmatrix}2&2&6\\2&3&3\\-1&2&-12\end{bmatrix} \xrightarrow{C_2 \mapsto C_2 - C_1} \begin{bmatrix}2&0&6\\2&1&3\\-1&3&-12\end{bmatrix} \xrightarrow{C_3 \mapsto C_3 - 3C_1} \begin{bmatrix}2&0&0\\2&1&-3\\-1&3&-9\end{bmatrix} \xrightarrow{C_3 \mapsto C_3 + 3C_2} \begin{bmatrix}2&0&0\\2&1&0\\-1&3&0\end{bmatrix}.$$

The notation $C_2 \mapsto C_2 - C_1$ says replace column 2 by column 2 minus column 1.

Slogan:

Column operations preserve the span of the columns.

5 Basis

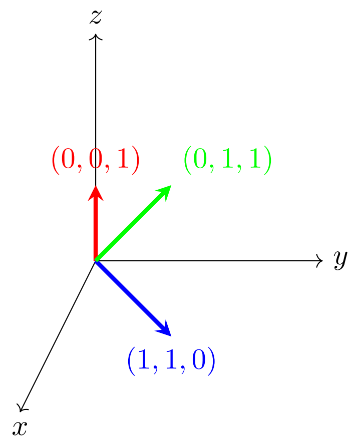
Suppose you are at the origin of $\mathbb{R}^n$, but I don't tell you what n is, so you don't even know what the dimension of the space you are exploring is. I tell you that your spaceship with 4 engines is able to explore all of the space (i.e. the 4 vectors span $\mathbb{R}^n$). Can you deduce that $n = 4$?

Actually no. Your engines might not work independently so you can only deduce that your space has dimension ≤ 4. For example, it could happen that $n = 3$ and your 4 engines were

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

This satisfies the condition that our engines can explore the entire space. The problem was the linear dependence. To deduce $n = 4$, I also need to tell you that none of your engines are redundant, i.e. the set of vectors on your spaceship forms a **basis**.

Definition 5.1 (basis)**.** A **basis** of a vector space V is a set of vectors that is both linearly independent and spans V.



Remark 5.2. The picture of a basis you should have in mind is a coordinate frame. For example,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$. Basis vectors do not need to be at right angles with each other. Actually, for general vector spaces, the concept of "angle" isn't even defined.

The standard basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is also basis for $\mathbb{R}^3$. A basis for $\mathbb{R}^3$ is basically any 3 vectors that are not coplanar (i.e. the vectors cannot all lie in the same plane).

Remark 5.3. If you're wondering whether basis has got anything to do with dimensions, the answer is yes. In $\mathbb{R}^n$, the number of vectors in a basis will always be n. You cannot get to the whole space with fewer than n vectors, and you cannot make your vectors linearly independent if you have $n + 1$ vectors.

Remark 5.4. There is an equivalent definition for a basis. The set $\{v_1, ..., v_n\}$ is a basis for V if every vector v can be expressed uniquely as a linear combination of $v_1, ..., v_n$. In terms of the spaceship analogy, to get to each point in space from the origin, there is exactly one way (not zero, not two) to do so (touching each joystick once).