Poisson's Equation

Mr. Poisson is swimming in his tank and would like to work out the equilbrium temperature of the water. To do this, he will use Poisson's Equation.

Poisson's Equation is

$$\begin{cases} -\nabla^2 u = f & \text{in } V, \\ u = g & \text{on } \partial V, \end{cases}$$

where V is a volume, and $f\colon V \to \mathbb{R}$ and $g\colon \partial V \to \mathbb{R}$ are given functions. The keen-eyed readers will notice a minus sign in front of ∇^2, and yes, that is included for better physical interpretation.

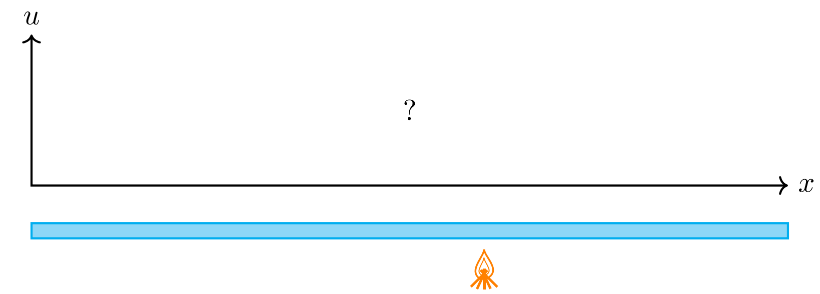
Poisson's equation can be used to model the following situation: given a heat source f, find the equilibrium temperature u of the region. We begin with a 1D example.

1 Poisson's Equation in 1 dimension

Example 1.1. Being slightly primitive, Mr. Poisson begins by modelling his tank as a 1 dimensional object. He lives in a rather long, straight tube, so a 1-dimensional approximation seems acceptable.

The tank is modelled as an interval $[a, b]$ such that the ends are kept at temperature T_0 by his owner. A heater is located at $x_0 \in (a, b)$ to keep the cold-blooded animal happy. The heat source can be represented by the function $f(x) = \delta(x - x_0)$ (depicted by the fire in the diagram below). The equilibrium temperature can be found by solving

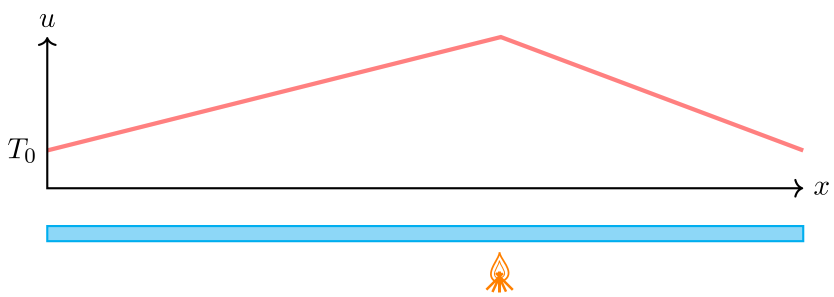
$$\begin{cases} -u''(x) = \delta(x - x_0) & x \in (a, b), \\ u(x) = T_0 & x \in \{a, b\}. \end{cases}$$



By solving the differential equation, we get

$$u(x) = \begin{cases} \frac{b-x_0}{b-a}(x - a) + T_0 & x \in [a, x_0) \\ -\frac{x_0-a}{b-a}(x - b) + T_0 & x \in (x_0, b]. \end{cases}$$

The graph of u is:



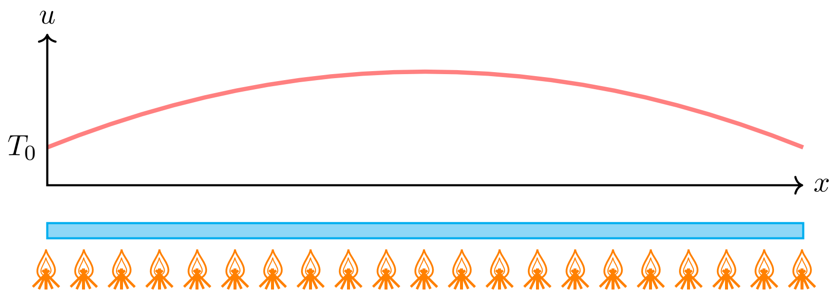
Actually, if we didn't need the exact equation, we could have sketched this solution from inspection: u'' is 0 for $x \neq x_0$, so u is a straight line on the interval (a, x_0) and (x_0, b). By integrating the equation up once, u' is a step function, and in fact at x_0 there is a jump of -1: $[u'(x)]_{x_0^-}^{x_0^+} = -1$.

In terms of the physical model of temperature, this makes sense: the tank is the hottest at where the heater is positioned, and heat is being drawn out at the boundary where the temperature is kept at T_0. Apart from the point where the heater is located, temperature drops off linearly.

Example 1.2. Same tank, but heating it uniformly. The equilibrium temperature satisfies

$$\begin{cases} -u''(x) = 1 & x \in (a, b), \\ u(x) = T_0 & x \in \{a, b\}. \end{cases}$$

which has the solution being a parabola:



This makes sense, with the tank being hottest in the middle. Don't worry about the fire getting close to the edge; the tank will always have temperature T_0 there. Also, don't worry about Mr. Poisson being fried; the temperature is adjusted to his tropical taste.

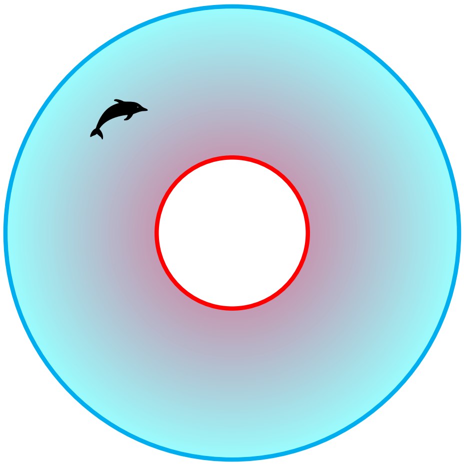
Remark 1.3. Although we are solving an *equilibrium* solution, there is still a time dimension, but the temperature stays constant over time. Heat enters from the heater and leaves from the edge, but this flow is constant, so the temperature is constant. It's like a thermostat trying to keep your house at the same temperature: cold near the windows, warmest in the middle of the living room or the kitchen, but all this time heat is lost.

Remark 1.4. Recall we said putting the minus sign in front of the Laplacian operator allows for better physical interpretation. This is because we want a positive f to correspond to a *heat* source. Thus it makes sense to call $-\nabla^2$ the *heat operator*. If we just had $\nabla^2 u = f$, then $f = \delta(x - x_0)$ would correspond to a *cold* source.

Remark 1.5. A *harmonic function* is a function satisfying $\nabla^2 u = 0$. In 1D, harmonic functions are just straight lines (boring!), but in higher-dimensions, they look more interesting. Promise!

2 Poisson's Equation in 2 dimensions

Example 2.1. Mr. Poisson now understands how the 1D heat equation behaves, and is ready to upgrade to 2D. Here we have a tank in the shape of an annulus (the area between two concentric circles). It makes a lovely aquarium as people have a full 360 degree view of his magnificent tank. Brave volunteers can even poke their heads in the middle circle for the full experience.



For a constant $a > 0$ and $u\colon \mathbb{R}^2 \to \mathbb{R}$, solve

$$\begin{cases} -\nabla^2 u = 0 & a \leq r \leq 2a \\ u = 1 & r = a \\ u = 0 & r = 2a. \end{cases}$$

Here, the boundary conditions say that the wall at $r = 1$ is kept at a constant temperature of 1 and the outer wall at a temperature of 0. (If you are worried about temperature being zero, just use Celsius. Here we are just using 0 for convenience.) This time, there are no heat sources within the tank; all the heat is provided by the inner wall. Thus we are looking for a harmonic function that satisfies the boundary conditions.

Note that I've persistently kept the minus sign in front of ∇^2 even though it makes no difference (the RHS is 0), because I insist that the minus sign goes with the Laplacian operator.

By symmetry, u is radially symmetric, so in fact u is a function of r only. The Laplacian in polar coordinates in 2D is

$$\nabla^2 u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right).$$

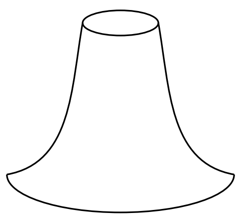
We obtain the solution to $-\nabla^2 u = 0$ by integrating:

$$\begin{aligned} -\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) &= 0 \\ r\frac{du}{dr} &= c \\ \frac{du}{dr} &= \frac{c}{r} \\ u(r) &= c\log(r) + b. \end{aligned}$$

By using $u(a) = 1, u(2a) = 0$, we get

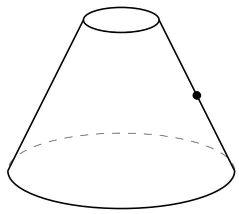
$$u(r) = -\log_2(r) + \log_2(a) + 1.$$

The graph looks like



where we've plotted the z-height as temperature. The tank is hottest on the inner wall and coldest on the outer wall, but the temperature does not decrease linearly, unlike the 1D example.

Remark 2.2. A common misconception is that harmonic functions are obtained by stretching a rubber sheet across the boundary to match the boundary conditions. In this case, we would obtain a truncated cone, also called a frustrum. This is not correct, as we saw above that the temperature decreases like log.



We can argue why temperature cannot decrease linearly with r geometrically. Suppose for contradiction that u is of the form $u(r) = mr + b$.

Recall the meaning of the second partial derivative:

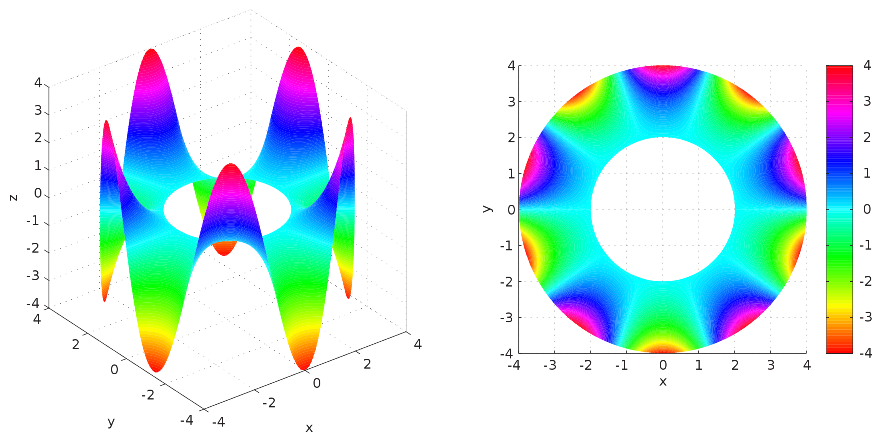
$$\frac{\partial^2 u}{\partial x^2} \geq 0 \iff u \text{ convex in the } x \text{ direction.}$$

Since u satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

the function must be convex in the x direction and concave in the y direction, or vice versa, as one term has to be negative of the other. In other words, the function has saddles everywhere. Now at the point marked on the frustrum in the diagram, the convexity is 0 in the left-right direction (it just slides linearly down the frustrum), but negative convexity in the front-back direction, so the sum of its convexities is negative and not zero, a contradiction.

Example 2.3. Wikipedia[1] provides another example. Suppose the inner wall has temperature 0 and the outer wall has a sinusoidal heating. Then the heat distribution looks like this.



This graph does look like it's a rubber sheet stretched between the inner and outer wall. But the equation for the rubber sheet is different; that solves the minimal surface equation, which is vastly ghastlier:

$$\left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right] \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right] \frac{\partial^2 u}{\partial x^2} = 0.$$

[1] https://en.wikipedia.org/wiki/Laplace%27s_equation

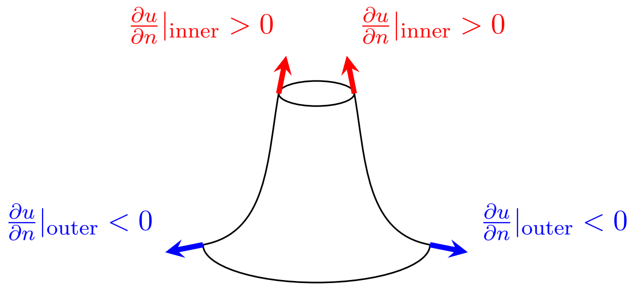
3 Directional Derivatives

When we have a boundary ∂V, we define the directional derivative of u in the direction normal to the boundary to be

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}), \qquad \mathbf{x} \in \partial V.$$

Note that both the normal and ∇u depend on the position $\mathbf{x}$ on the boundary. The result $\frac{\partial u}{\partial n}$ is a scalar, hence the alternative notation $\frac{\partial u}{\partial \mathbf{n}}$ can be slightly misleading.

In the annulus tank, we can see that $\frac{\partial u}{\partial n}$ at the outer wall is negative, as the function slopes downwards there.



For the inner wall, note that the normal is pointing towards the origin, because by convention the normal to a region points away from the region. Hence standing near the inner wall and looking towards the origin, we see the temperature rising, hence $\frac{\partial u}{\partial n} > 0$ there. Diagrammatically, the vectors I've drawn are

$$\left(\mathbf{n}(\mathbf{x}), \frac{\partial u}{\partial n}(\mathbf{x})\right),$$

where the xy-coordinate of the 3D vector represents the direction of $\mathbf{n}$ and the z-value represents increase or decrease of temperature.

4 Boundary Conditions

There are two main types of boundary conditions.

- **Dirichlet Boundary Conditions.** Solve

$$\begin{cases} -\nabla^2 u = f & \text{in } V, \\ u = g & \text{on } \partial V, \end{cases}$$

  which we regard as specifying the temperature at the walls.

- **Neumann Boundary Conditions.** Solve

$$\begin{cases} -\nabla^2 u = f & \text{in } V, \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial V, \end{cases}$$

  which we regard as specifying the amount of heat we pump in or take out from the walls. In particular,

$$\begin{aligned} \frac{\partial u}{\partial n} > 0 \text{ on } \partial V &\implies \text{we are pumping heat in} \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial V &\implies \text{wall is an insulator} \\ \frac{\partial u}{\partial n} < 0 \text{ on } \partial V &\implies \text{we are pumping heat out.} \end{aligned}$$

5 Electrostatics

Earlier, I pre-empted the concern with zero temperature. Heat distribution is a positive quantity, so the examples above with zero boundary conditions were not completely natural. There is also a problem with negative sources ($f < 0$), which annihilate positive sources ($f > 0$). A more realistic physical example of Poisson's Equation is in electrostatics, where f is the charge distribution and u is the electric potential. In fact, this is more often written as Gauss' law:

$$-\nabla^2 \varphi(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0},$$

where $\mathbf{x} \in \mathbb{R}^3$, $\varphi(\mathbf{x})$ is the electric potential, $\rho(\mathbf{x})$ is the charge distribution, and ε_0 is a constant. A negative heat source makes sense – it is just the electric potential due to a negative charge.

To spell it out more clearly, we can replace

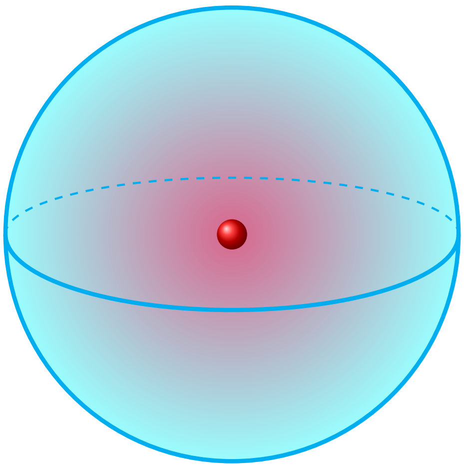
$$\begin{aligned} \text{temperature} &\leftrightarrow \text{voltage} \\ \text{heat source} &\leftrightarrow \text{positive charge} \\ \text{cold source} &\leftrightarrow \text{negative charge.} \end{aligned}$$

The Dirichlet boundary condition would be requiring the wall to be at a certain voltage.

There are other physical situations that can affect the boundary condition. A conductor filling the region $D \subseteq \mathbb{R}^3$ is requiring φ to be constant in D. Surface charges on ∂V corresponds to a jump condition in $\frac{\partial \varphi}{\partial n}$.

Example 5.1. The diagram below is a picture of the voltage due to a positive point charge at the origin in $\mathbb{R}^3$ sitting inside a grounded sphere (zero voltage) of radius 1. Hence we have to solve the equation:

$$\begin{cases} -\nabla^2 \varphi = +\dfrac{Q}{\varepsilon_0}\delta(\mathbf{x}) & |\mathbf{x}| \leq 1 \\ u = 0 & |\mathbf{x}| = 1. \end{cases}$$



Let's take $Q = \varepsilon_0$ so that we don't have to carry the burdensome constant of Q/ε_0 around:

$$\begin{cases} -\nabla^2 \varphi = \delta(\mathbf{x}) & |\mathbf{x}| \leq 1 \\ u = 0 & |\mathbf{x}| = 1. \end{cases}$$

By symmetry, φ is a function of r only. The idea is to use the divergence theorem. Let $S = B_r(\mathbf{0})$, the sphere of radius r around the origin. Then by computing the total charge in V:

$$1 = \int_V \delta(\mathbf{x})\, dV$$

$$
\begin{aligned}
&= -\int_V \nabla^2\varphi \, dV && \text{Poisson's equation} \\
&= -\int_{\partial V} \nabla\varphi \cdot d\mathbf{S} && \text{divergence theorem in 3D} \\
&= -\int_{\partial V} \nabla\varphi \cdot \mathbf{e}_r \, dS && \text{boundary of sphere has normal pointing outwards} \\
&= -\int_{\partial V} \varphi'(r) \, dS && \varphi \text{ is a function of } r \\
&= -4\pi r^2 \varphi'(r) && \text{surface area of sphere.}
\end{aligned}
$$

Hence the solution is

$$\varphi'(r) = -\frac{1}{4\pi r^2} \implies \varphi(r) = \frac{1}{4\pi r} + c.$$

Now $\varphi(1) = 0$, so the solution is

$$\varphi(r) = \frac{1}{4\pi}\left(\frac{1}{r} - 1\right).$$

which agrees with the picture where the potential is higher near the positive charge and 0 at $r = 1$.

Incidentally, the exact same equation can be used to model a point heat source at the origin surrounded by a sphere kept at temperature 0! This is because, as we've said, both temperature and electric potential satisfies Poisson's equation. Hence this equation is very useful for an electric eel who is trying to determine both.

6 Average Convexity

Here's another way to understand the quantity ∇^2. We saw that

$$\nabla^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the sum of convexities in all of the directions. Up to a scale factor of $\frac{1}{n}$, this quantity represents the convexity at a point averaged in all n directions.

$$
\begin{aligned}
\nabla^2 u(\mathbf{x}) > 0 &\implies \mathbf{x} \text{ is more convex than concave,} \\
\nabla^2 u(\mathbf{x}) = 0 &\implies \mathbf{x} \text{ has convexity and concavity balancing out,} \\
\nabla^2 u(\mathbf{x}) < 0 &\implies \mathbf{x} \text{ is more concave than convex.}
\end{aligned}
$$

So ∇^2 is one way to generalise the notion of "second derivatives" in higher dimensional space. The difference between the Laplacian and the Hessian is that the Laplacian has "averaged out" the convexities, whereas the Hessian still encodes all the information about which direction is convex and which is concave.

7 Mean Value Property

Mr. Poisson is located at $\mathbf{x}$ and knows that the value of $\nabla^2 u(\mathbf{x})$ is zero. What information does this tell him? From the previous section, he knows that the graph has average convexity zero. What can he say about the temperature of the surrounding waters? We refer him to the mean value property of harmonic functions.

Theorem 7.1 (Mean Value Property)**.** Let $u\colon V \to \mathbb{R}$ be harmonic, $\mathbf{x}_0 \in V$, and $r > 0$ be small enough that the ball of radius r centred at $\mathbf{x}_0$ (denoted $B_r(\mathbf{x}_0)$) is still contained in V. Then

$$u(\mathbf{x}_0) = \frac{1}{4\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u \, dS.$$

Here

$$\begin{aligned} B_r(\mathbf{x}_0) &= \{\mathbf{y} : |\mathbf{y} - \mathbf{x}_0| \le r\} \\ \partial B_r(\mathbf{x}_0) &= \{\mathbf{y} : |\mathbf{y} - \mathbf{x}_0| = r\}. \end{aligned}$$

Proof. WLOG $\mathbf{x}_0$ is the origin. Let F be the average value on a sphere of radius r:

$$\begin{aligned} F(r) &= \frac{1}{4\pi r^2} \int_{S_r} u(\mathbf{x}) \, dS \\ &= \frac{1}{4\pi r^2} \int_0^\pi \int_0^{2\pi} u(\mathbf{x}) \, r^2 \sin\theta \, d\theta \, d\phi \\ &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(\mathbf{x}) \sin\theta \, d\theta \, d\phi. \end{aligned}$$

Take the r-derivative:

$$\begin{aligned} F'(r) &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r}(\mathbf{x}) \sin\theta \, d\theta \, d\phi && \text{diff. under integral sign} \\ &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \nabla u \cdot \mathbf{e}_r \sin\theta \, d\theta \, d\phi && \text{defn of directional derivative} \\ &= \frac{1}{4\pi r^2} \int_0^\pi \int_0^{2\pi} \nabla u \cdot \mathbf{e}_r r^2 \sin\theta \, d\theta \, d\phi && \text{restore the term } r^2 \\ &= \frac{1}{4\pi r^2} \int_{S_r} \nabla u \cdot d\mathbf{S} && \text{put } d\mathbf{S} \text{ together} \\ &= \frac{1}{4\pi r^2} \int_{V_r} \nabla^2 u \, dV && \text{divergence theorem} \\ &= 0. \end{aligned}$$

So F is constant in r, and hence the average value on a sphere of radius r around a point does not depend on the radius r. Now shrink $r \to 0$. □

Remark 7.2. The main idea of the proof is that r is constant on S_r and hence we can take it out of the integral and cancel with the r in front, so that when we take the r-derivative it only affects u. Now put it back in to reconstruct the Jacobian.

Hence we have another way to think about $-\nabla^2$. The fact $-\nabla^2 u = 0$ says that the average temperature of the surrounding waters is the same as where he is.

What about if $-\nabla^2 u > 0$? In this case, from the proof above which says,

$$F'(r) = \frac{1}{4\pi r^2} \int_{V_r} \nabla^2 u \, dV,$$

we have $F'(r) < 0$, meaning as Mr. Poisson moves away from the centre, the temperature of the sphere drops. This makes sense, as $-\nabla^2 u = f > 0$ means the forcing there is a heat source, so the centre should be hotter than the neighbour. Similarly for a cold source. So $-\nabla^2$ measures the temperature difference between a point $\mathbf{x}$ and its neighbours:

$$\begin{cases} -\nabla^2 \varphi < 0 \implies \text{neighbours are colder than the centre} \\ -\nabla^2 \varphi = 0 \implies \text{neighbours have the same temperature as the centre} \\ -\nabla^2 \varphi > 0 \implies \text{neighbours are hotter than the centre.} \end{cases}$$

Remark 7.3. The consequence of the mean-value property is that no interior point of a harmonic function can be a maximum or a minimum (unless the function is constant). Intuitively, if there is no heat or cold source, the temperature cannot have a local maximum in the interior, otherwise the temperature would not be in equilibrium, and the temperature would decrease.

8 Heat Equation

To model a non-equilibrium heat distribution, we use the PDE called the heat equation (or the diffusion equation):

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) - D\nabla^2 u(\mathbf{x}, t) = f(\mathbf{x}, t),$$

with initial conditions $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, and certain boundary conditions. $D > 0$ is the diffusion coefficient, which controls how fast heat spreads. Note we recover Poisson's equation if we set $\frac{\partial u}{\partial t}(x, t) = 0$, i.e. heat is not changing in time.

Consider the heat distribution u with no forcing term ($f = 0$), so

$$\frac{\partial u}{\partial t} = \nabla^2 u.$$

- If $\nabla^2 u(\mathbf{x}, t) > 0$, i.e. the function is more convex than concave, then $\frac{\partial u}{\partial t} > 0$, i.e. temperature will increase there.
- If $\nabla^2 u(\mathbf{x}, t) < 0$, i.e. the function is more concave than convex, then $\frac{\partial u}{\partial t} < 0$, i.e. temperature will decrease there.
- If $\nabla^2 u(\mathbf{x}, t) = 0$, i.e. function has convexity 0, then the temperature is unchanged.

What the Heat Equation says is that the temperature at a point $\frac{\partial u}{\partial t}$ will rise if its neighbours are hotter, and temperature will drop if its neighbours are colder.

Example 8.1. The solution to

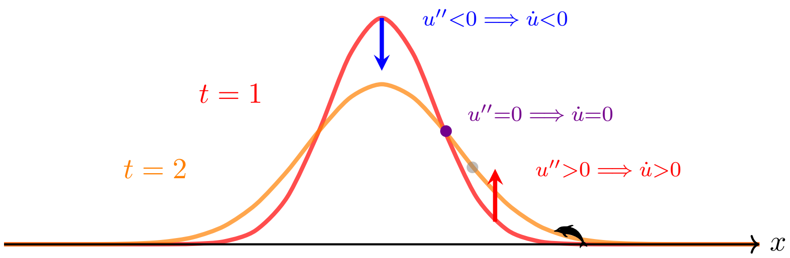
$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t) - D\dfrac{\partial^2 u}{\partial x^2}(x, t) = 0 \\ u(x, 1) = \dfrac{1}{\sqrt{4\pi D}} e^{-x^2/4D} \end{cases}$$

is

$$u(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}.$$

This models an initial bell-shaped heat distribution with diffusion coefficient $D > 0$ in an infinitely long 1-dimensional tank.

We will not explain how we obtained this solution, but let's look at its properties, and see if it agrees with our intuition of "heat". Here's a picture:



The peak clearly has negative convexity, so as the rate of change of temperature is proportional to the temperature, the temperature should drop. Far away from the peak, the temperature should rise, as the peak heats up its surroundings. There is a point at which the temperature stays still, which is the inflection point. You might worry that this inflection point moves, and in fact it does, as shown by the faded gray dot, so let's rephrase the previous sentence to say the temperature *temporarily* stays still (has zero rate of change, if you prefer this phrasing).

If Mr. Poisson is located at the position shown in the diagram, then he would experience a rise in temperature and then a drop in temperature as the heat moves out to infinity. The equilibrium temperature is the constant zero solution.

9 Flattest surface

As a homework exercise, you were asked to show

Theorem 9.1. Let u be harmonic in V, and w a scalar field which satisfies $w = u$ on ∂V. Show that

$$\int_V |\nabla w|^2 \, dV \geq \int_V |\nabla u|^2 \, dV.$$

We interpret the quantity

$$\int_V |\nabla u|^2 \, dV$$

as the *total steepness* of the function in the volume. Note that we are integrating over $|\nabla u|^2$ rather than just $|\nabla u$. This is very common in mathematics, as quadratics are much nicer than the absolute value function, because it is differentiable; e.g. in probability, the variance, which measures the spread of a random variable from its mean, averages over the *square* distance from its mean. Hence what the theorem says is that

harmonic functions are the flattest possible functions, subject to the boundary conditions!

This is consistent with the results before about the mean-value property. Note "flattest" means the most horizontal, and is not the same as "minimal surface area".

10 Existence and Uniqueness*

This section is intended as a bonus section.

Existence means there is *at least one* solution,
Uniqueness means there is *at most one* solution.

Example 10.1. Let's show that Poisson's equation with Neumann Boundary Conditions has uniqueness but not necessarily existence.

Uniqueness

Suppose u_1 and u_2 are solutions. Then $\varphi = u_1 - u_2$ satisfies

$$\begin{cases} -\nabla^2 \varphi = -\nabla^2 u_1 + \nabla^2 u_2 = f - f = 0 & \text{in } V, \\ \dfrac{\partial \varphi}{\partial n} = \dfrac{\partial u_1}{\partial n} - \dfrac{\partial u_2}{\partial n} = g - g = 0 & \text{on } \partial V, \end{cases}$$

and consider the sum of the steepness of φ:

$$\begin{aligned}
0 &\leq \int_V |\nabla\varphi|^2 \, dV \\
&= \int_V \nabla\varphi \cdot \nabla\varphi \, dV \\
&= \int_V \nabla \cdot (\varphi\nabla\varphi) - \varphi\cancelto{0}{\nabla^2\varphi} \, dV && \text{product rule} \\
&= \int_{\partial V} \varphi\nabla\varphi \cdot d\mathbf{S} && \text{divergence theorem} \\
&= \int_{\partial V} \varphi \cancelto{0}{\frac{\partial\varphi}{\partial n}} \, dS \\
&= 0.
\end{aligned}$$

Since LHS and RHS are both 0, the inequality must in fact be an equality, so $|\nabla\varphi|^2$ is identically 0 in V, so φ is constant in V. Because $\varphi = u_1 - u_2$ is zero on the boundary, this constant is 0. Hence $u_1 = u_2$ in V.

The idea behind the proof was to apply the divergence theorem somehow so we can pass the information from the interior of the volume to the boundary.

Non-existence

Suppose $f = 0$ (no heat source) and $g = 1$ (pumping in heat) in a bounded volume. Then intuitively, there is no *equilibrium* temperature because the room is just going to heat up exponentially, hence no solution. We can in fact take a 1D counter-example.

$$\begin{cases} -u'' = 0 & x \in (a, b) \\ \dfrac{\partial u}{\partial n} = 1 & x \in \{a, b\}. \end{cases}$$

We have

$$\frac{\partial u}{\partial n}(b) = u'(b) = 1,$$

but the normal at a is pointing in the $-x$ direction due to pointing *away* from the interval $[a, b]$, so

$$\frac{\partial u}{\partial n}(a) = \frac{\partial u}{\partial(-x)}(a) = -u'(a) = -1.$$

This is consistent with the heat analogy that if we are pumping heat in, the wall should be hotter than the interior.

This clearly does not have a solution: we cannot have a straight line that has slope -1 at a and slope 1 at b.