Part III — Metric Embeddings

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Functional Analysis and Part II Probability and Measure are essential

Definitions, basic examples and motivations. Frechét embeddings, Aharoni's theorem (ℓ_∞, c_0), Euclidean distortion, Bourgain's embedding theorem (ℓ_2, L_2). Obstructions to embeddings. Poincaré's inequalities (L_1, L_2). Dimension reduction in L_2 (Johnson-Lindenstrauss Lemma). Lack of dimension reduction in L_1. Local theory of Banach spaces, Ribe programme. Bourgain's characterisation of super-reflexivity, metric type and cotype and/or metric Dvoretzky's Theorem). Coarse embeddings of ℓ_2 into Banach spaces, coarse embeddings into uniformly convex/uniformly smooth Banach spaces.

Books: Ostrowski's *Metric Embeddings*, Matousek's *Lectures in discrete geometry* (Ch15 - extended online notes), *Lectures in metric embeddings* (available online). Assaf Naor's survey article on the Ribe programme.

Related Part III courses: discrete analysis of Fourier series, some combinatorics.

Contents

1 Basic Definitions, Examples and Motivations

Definition. A *metric space* is a set M with a metric $d\colon M \times M \to \mathbb{R}$ such that (i) $d(x, y) \geq 0$ for all x, y, $d(x, x) = 0$ for all x, (ii) $d(x, y) = d(y, x)$ (symmetry), (iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality), (iv) $d(x, y) = 0 \implies x = y$. If d satisfies (i),(ii) and (iii) then it's a *semimetric*.

Example. Graph with the graph distance. A *graph* is a pair (V, E) where V is a set and $E \subset V^{(2)} = \{e \subset V : |e| = 2\}$. Elements of V are the *vertices* of G and elements of E are the *edges* of G. A *walk* in G is a sequence $x_0, x_1, ..., x_n$ of vertices such that $x_{i-1}x_i \in E$ for all $1 \leq i \leq n$. [Given $e = \{x, y\} \in E$, x, y are the *endvertices* of e, write $e = xy = yx$. We also write $x \sim y$]. The length of the walk is n. This is called a *walk from x_0 to x_n*. If $x_i \neq x_j$ whenever $1 < j - i < n$, this walk is called a *path* from x_0 to x_n. Say G is *connected* if for all $x, y \in V$ there exists a walk (or a path) in G from x to y. The *graph distance* is defined as $d_G(x, y) =$ the length of a shortest path in G from x to y. Some standard graphs: K_n is the *complete graph on n vertices*, all $\binom{n}{2}$ edges are present. Here

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

P_n is the *path of length n*, with $n + 1$ vertices $x_0, x_1, ..., x_n$, and $E = \{x_{i-1}x_i : 1 \leq i \leq n\}$. As a metric space, this is $\{0, 1, ..., n\}$ with $d(x, y) = |x - y|$. C_n is the *cycle of length n*. $V = \{x_1, ..., x_n\}$ and $E = \{x_i x_{i+1} : 1 \leq i < n\} \cup \{x_1 x_n\}$. B_n is the *rooted binary tree of depth n*. And finally, H_n is the *Hamming cube* $V = \{0, 1\}^n$, $x \sim y$ iff there exists exactly one coordinate i such that $x_i \neq y_i$. Then $d(x, y) = |\{i : x_i \neq y_i\}|$.

Example. Groups with the word metric. Let G be a group generated by some subset S. We always assume that $e \notin S$ and S is symmetric: $\forall x \in S$, $x^{-1} \in S$. The *word metric* is defined to be $d(x, y) = \min\{n : \exists a_1, ..., a_n \in S \, s.t. \, x^{-1}y = a_1...a_n\}$. The *Cayley Graph* $C(G, S)$ has vertex set G and $x \sim y$ iff $x^{-1}y \in S$. The graph distance on G is d.

Example. Cut semimetrics. A *cut* on a set M is a partition of M into S and $M \setminus S$. The corresponding *cut semimetric* is

$$d_S(x, y) = \begin{cases} 0 & x, y \text{ are in the same part} \\ 1 & \text{otherwise.} \end{cases}$$

Definition. A *normed space* is a real or complex vector space V with a *norm* on V, i.e. a function $\|\cdot\|\colon V \to \mathbb{R}$ such that (i) $\|x\| \geq 0$ for all $x \in V$, (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all λ scalars and $x \in V$, (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$, (iv) $\|x\| = 0 \implies x = 0$. Then $d(x, y) = \|x - y\|$ is a metric on V. If V is complete then it is called a *Banach space*. If $\|\cdot\|$ satisfies (i),(ii) and (iii) then it is called a *seminorm*.

Example. Classical sequence spaces.

- $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ for $1 \leq p \leq +\infty$, with $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. Here e_i is the standard ith basis vector. If $p = \infty$ the norm is $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

- $\ell_p = \{(x_i)_{i=1}^{\infty} : \sum_{i=1}^{n} |x_i|^p\}$ for $1 \le p < +\infty$, with $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$. $\ell_\infty = \left\{(x_i)_{i=1}^{\infty} : \sup_{i\ge 1} |x_i| < \infty\right\}$, with $\|x\|_\infty = \sup_{i\ge 1} |x_i|$. More generally, for a set S, $\ell_\infty(S) = \{x \colon S \to \mathbb{R} : x \text{ is bounded}\}$. The norm is $\|x\|_\infty = \sup_{s\in S} |x(s)|$. Note $c_0 = (x_i)_{i=1}^{\infty} : x_i \to 0\}$ is a closed subspace of ℓ_∞.

Example. Classical function spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

- For $1 \le p < \infty$, $L_p(\mu) = \{f \colon \Omega \to \mathbb{R} : f \text{ measurable}, \int_\Omega |f|^p \, d\mu < \infty\}$ equipped with $\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$.
- For $p = \infty$, $L_\infty(\mu) = \{f \colon \Omega \to \mathbb{R} : f \text{ measurable}, \exists N \in \mathcal{F}, \mu(N) = 0, f \text{ bounded on } \Omega \backslash N\}$, equipped with $\|f\|_\infty = \operatorname{ess\,sup}(f) = \inf\{\sup_{\Omega\backslash N} |f| : N \in \mathcal{F}, \mu(N) = 0\}$.
- In the case $\Omega = [0,1]$, μ= Lebesgue measure, we write L_p for $L_p(\mu)$.
- For compact space K, $C(K) = \{f \colon K \to \mathbb{R} : f \text{ cts}\}$ is a closed subspace of $\ell_\infty(K)$, e.g. $C([0,1])$.

Example. Hilbert Space. An *inner product space* (IPS) is a vector space V equipped with an inner product $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$ (symmetric bilinear, positive definite). Then V becomes a normed space with $\|x\| = \langle x, x \rangle^{1/2}$. If V is complete wrt $\|\cdot\|$, then it's called a *Hilbert space*.

Definition. Let $f \colon M \to N$ be a map between metric spaces. Then f is *isometric* or an *isometric embedding* if $d(f(x), f(y)) = d(x,y)$ for all $x, y \in M$. We say f is a *bilipschitz embedding* if $\exists a, b > 0$ such that

$$a d(x,y) \le d(f(x), f(y)) \le b d(x,y) \qquad \forall x, y \in M. \tag{1}$$

The *distortion* of f is $\operatorname{dist}(f) = \min\{\frac{b}{a} : (1) \text{ holds for } f\}$.

Remark.

(i) If $a = b$, then f is a *scaled isometric embedding.*

(ii) Definition makes sense for semimetrics.

(iii) If (1) holds, then f is Lipschitz with Lipschitz constant $\operatorname{Lip}(f) \le b$, where

$$\operatorname{Lip}(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x,y)} : x, y \in M, x \ne y\right\}.$$

Also f is injective (because of the LH inequality) and $f^{-1} \colon f(M) \to M$ is Lipschitz, with $\operatorname{Lip}(f^{-1}) \le \frac{1}{a}$. Then $\operatorname{dist}(f) = \operatorname{Lip}(f)\operatorname{Lip}(f^{-1})$.

Recall, if $T \colon X \to Y$ is a linear map between normed spaces, then T is continuous iff T is bounded ($\exists C > 0, \|Tx\| \le C\|x\|$ for all $x \in X$). The smallest C is $\|T\|$ iff T is Lipschitz, $\|T\| = \operatorname{Lip}(T)$. T is an isomorphism if T is a bijection, both T and T^{-1} are bounded. T is an *isometric embedding* or *into isomorphism* if T is an isomorphism between X and $T(X)$, iff T is bilipschitz. Then $\operatorname{dist}(T) = \|T\|\left\|T^{-1}\right\|$. T is an *isometric isomorphism embedding* if $\|Tx\| = \|x\|$ for all $x \in X$.

Notation. Write $X \hookrightarrow_C Y$ if there exists an isomorphism embedding $T\colon X \to Y$ with $\|T\| \|T^{-1}\| \leq C$. We say X *C-embeds* into Y. So $X \hookrightarrow_1 Y$ iff there exists an isometric isomorphic embedding $X \to Y$. $X \sim Y$ means X, Y are isomorphic. $X \cong Y$ means X, Y are isometrically isomorphic.

Example. (i) $\ell_p^n \hookrightarrow_1 \ell_p$.

(ii) $\ell_p \hookrightarrow_1 L_p = L_p([0,1], \lambda = \text{Leb})$. *proof:* Fix pairwise disjoint measurable sets $(A_i)_{i=1}^\infty$ each of positive measure. For $1 \leq p < \infty$, consider

$$(x_i)_{i=1}^\infty \mapsto \sum_{i=1}^\infty x_i \mathbb{1}_{A_i} \lambda(A_i)^{-1/p},$$

and for $p = \infty$, consider $(x_i)_{i=1}^\infty \mapsto \sum_{i=1}^\infty x_i \mathbb{1}_{A_i}$.

Fact. If (Ω, μ) is a measure space, $X \subset L_p(\Omega, \mu)$ separable, then $X \hookrightarrow_1 L_p$.

Notation. For a normed space X, let $B_X = \{x \in X : \|x\| \leq 1\}$ be the *closed unit ball*, and $S_X = \{x \in X : \|x\| = 1\}$, the *unit sphere* of X.

Proposition 1.1. For all $n \in \mathbb{N}$, $\ell_2^n \hookrightarrow_1 L_p$ for any $1 \leq p \leq \infty$.

Proof. **Case $1 \leq p < \infty$.** Let $B = B_{\ell_2^n}$, $\mu =$ Lebesgue measure on B, $S = S_{\ell_2^n}$. Since μ is rotation invariant, the value of $\int_B |\langle x, \omega\rangle|^p \, d\mu(\omega)$ is the same for all $x \in S$. Call this α. Define $T\colon \ell_2^n \to L_p(B, \mu)$ by $(Tx)(\omega) = \langle x, \omega \rangle \alpha^{-1/p}$. Then T is linear and $\|Tx\|_p^p = \int_B |\langle x, \omega\rangle|^p \alpha \, d\mu(\omega) = \|x\|_2^p$ for all $x \in \ell_2^n$. To finish, use the fact above to embed $L_p(B, \mu) \hookrightarrow_1 L_p$.

Case $p = \infty$. This follows from the next result and example above. □

Definition. Let X be a normed space. The *dual space* X^* of X is $X^* = \mathcal{B}(X, \mathbb{R}) = \{f\colon X \to \mathbb{R} : f \text{ linear bounded}\}$. The operator norm is $\|f\| = \sup\{|f(x)| : x \in B_X\}$. By the Hahn-Banach theorem, $\forall x \in X$, $\exists f \in S_{X^*}$ such that $f(x) = \|x\|$. So $\|x\| = \max\{g(x) : g \in S_{X^*}\}$.

Proposition 1.2. Let X be a separable normed space. Then $X \hookrightarrow_1 \ell_\infty$.

Proof. Let $(x_n)_{n=1}^\infty$ be dense in X. Then for all $n \in \mathbb{N}$, choose $f_n \in S_{X^*}$, $f_n(x_n) = \|x_n\|$ by Hahn-Banach. Define $T\colon X \to \ell_\infty$ by $Tx = (f_n(x))_{n=1}^\infty$. Given $x \in X$, $|f_n(x)| \leq \|f_n\| \|x\| = \|x\|$ for all n. So T is well-defined. T is linear and T is bounded with $\|T\| \leq 1$. For all $n \in \mathbb{N}$, $\|Tx_n\| = \|x_n\|$. So T is isometric on a dense set, so by continuity T is isometric on the whole space X. □

Remark. For any normed space X there exists a set S such that $X \hookrightarrow_1 \ell_\infty(S)$, e.g. $S = S_{X^*}$.

Corollary 1.3. (Corollary to proposition 1.1) Let M be a finite metric space. If M embeds into L_2 with distortion $\leq D$, then M embeds into L_p for all $1 \leq p \leq \infty$ with distortion $\leq D$. i.e. L_2 is the hardest thing to embed into.

Proposition 1.4. If M is an n-element subset of $L_1(\Omega, \mu)$, then $M \hookrightarrow_1 \ell_1^N$, where $N = n!$.

Proof. Let $M = \{f_1, ..., f_n\}$. [Aside: $f_i \mapsto \int_\Omega f_i \, d\mu$ is an obvious $L_1(\Omega, \mu) \to \mathbb{R}$, but

$$\left| \int f_i - \int f_j \right| \le \int |f_i - f_j| = \|f_i - f_j\|_{L^1}$$

has equality if say $f_i \le f_j$ a.e.] There exists a partition $\Omega = \bigcup_{\pi \in S_n} \Omega_\pi$ of Ω, where $\Omega_\pi \subset \{\omega \in \Omega : f_{\pi(1)}(\omega) \le ... \le f_{\pi(n)}(\omega)\}$. Here we have used the finiteness of M. [Note that we have used the subset symbol. When two fs are equal for some ω, we can arbitrarily put it in just one of the Ω_πs.] Then

$$\|f_i - f_j\|_1 = \int_\Omega |f_i - f_j| \, d\mu = \sum_{\pi \in S_n} \int_{\Omega_\pi} |f_i - f_j| \, d\mu = \sum_{\pi \in S_n} \left| \int_{\Omega_\pi} f_i - \int_{\Omega_\pi} f_j \right|.$$

Define $T\colon M \to \ell_1^N$ by $Tf_i = \left(\int_{\Omega_\pi} f_i \, d\mu \right)_{\pi \in S_n}$. So above $= \|Tf_i - Tf_j\|_1$. □

Example (More examples)**.** (i) C_4 embeds bilipschitzly into ℓ_2^2 naturally, with distortion $\sqrt{2}$. It doesn't embed isometrically. In ℓ_2, $d(x, z) = d(x, y) + d(y, z)$ iff $y \in [x, z] = \{(1 - t)x + tz : 0 \le t \le 1\}$. It follows that ℓ_2 has the *unique midpoint property*: $\forall x, z \in \ell_2$ there is at most one point y (in fact exactly 1) such that $d(x, y) = d(y, z) = \frac{1}{2} d(x, z)$. C_4 does not have this property.

(ii) Any n-element set in a Hilbert space embeds isometrically into ℓ_2^{n-1}. Cannot do better in general. See example sheet. If we relax the condition to bilipschitz, then we can do much better. In fact, $\forall \epsilon 0, \exists c > 0$ any n-element set in Hilbert space embeds into ℓ_2^m where $m = c \log n$ with distortion $< 1 + \epsilon$. See later for proof.

Observe: If M is a finite metric space, N is a metric space and $|N| \ge |M|$, then M bilipschitzly embeds into N.

Definition. Given families $(M_\alpha)_{\alpha \in A}$ and $(N_\alpha)\alpha \in A$ of metric spaces, embeddings $f_\alpha\colon M_\alpha \to N_\alpha$, $\alpha \in A$, are called *uniformly bilipschitz* if $\sup_{\alpha \in A} \operatorname{dist}(f_\alpha) < \infty$.

The sparsest cut problem.

Let $G = (V, E)$ be a connected, finite graph. We are given two functions $C\colon E \to \mathbb{R}^+ = [0, \infty)$ (*capacity*) and $D\colon V \times V \to \mathbb{R}^+$ (*demand*). A *cut* of G is a partitioning $(S, V \setminus S)$ of V. The *capacity of* $(S, V \setminus S)$ is

$$C(S, V \setminus S) = \sum_{uv \in E, u \in S, v \notin S} C(uv).$$

The demand of the cut is

$$D(S, V \setminus S) = \sum_{u \in S, v \notin S} D(uv).$$

The *sparsity* of the cut is $C(S, V \setminus S)/D(S, V \setminus S)$ whenever $D(S, V \setminus S) \neq 0$. This is NP-hard. So we look at an equivalent problem: Minimise over all cuts with nonzero demand of the following quantity

$$\frac{\sum_{uv \in E} C(uv) d_S(u, v)}{\sum_{u, v \in V} D(u, v) d_S(u, v)}$$

where d_S is the cut semimetric. Note the denominator is twice $D(S, V \setminus S)$.

Let $\varphi^*(C, D)$ be this minimum. The idea is to minimise

$$\sum_{uv \in E} C(uv) d(u, v),$$

subject to d being a semimetric and $\sum_{u,v \in V} D(u, v) d(u, v) = 1$. This is now a linear programming problem with a linear normalisation condition. The property that d is a semimetric is just constraints with inequalities. There are fast algorithms to solve this.

Let $\varphi(C, D)$ be the minimum and d_{min} be a semimetric that achieves this minimum. Clearly $\varphi(C, D) \leq \varphi^*(C, D)$.

Lemma 1.5. Let (M, d) be a finite semimetric space. Then (M, d) embeds isometrically into L_1 iff d is a non-negative linear combination of cut semimetrics.

Proof. $(\Leftarrow)$ We assume there exists cuts $(S_i, M \setminus S_i)$ for $i = 1, ..., k$ and non-negative reals α_i, $i = 1, ..., k$, such that $d = \sum_{i=1}^k \alpha_i d_S$. Let $f_i \colon M \to \mathbb{R}$ be $f_i(x) = \alpha_i \mathbb{1}_{x \in S_i}$, and $f \colon M \to \ell_1^k$, $f(x) = (f_i(x))_{i=1}^k$. Then

$$\|f(x) - f(y)\|_1 = \sum_{i=1}^k |f_i(x) - f_i(y)| = \sum_{i=1}^k \alpha_i d_{S_i}(x, y) = d(x, y).$$

$(\Rightarrow)$ By proposition 4, there exists isometric embedding $f \colon M \to \ell_1^k$, some $k \in \mathbb{N}$. Enumerate $\{f(x)_i : x \in M\}$ as $\beta_{i1} < \beta_{i2} < ... < \beta_{im_i}$. Let $S_{ij} = \{x : f(x)_i \leq \beta_{ij}\}$, for $1 \leq i \leq k$, $1 \leq j \leq m_i$. Fix $x, y \in M$, and fix $1 \leq i \leq k$. Suppose $f(x)_i = \beta_{ij_1} \leq f(y)_i = \beta_{ij_2}$. $x \in S_{ij}$ for $j \leq j_1$, $y \leq S_{ij}$ for $j \leq j_2$. If we look at the sum

$$\begin{aligned} \sum_{j=1}^{m_i - 1} (\beta_{i,j} - \beta_{i,j-1}) d_{S_{ij}}(x, y) &= \sum_{j=j_1+1}^{j_2} (\beta_{i,j} - \beta_{i,j-1}) \\ &= \beta_{i,j_2} - \beta_{i,j_1} \\ &= f(y)_i - f(x)_i = |f(x)_i - f(y)_i|. \end{aligned}$$

Sum over i:

$$\sum_{i=1}^k \sum_{j=1}^{m_i - 1} (\beta_{i,j} - \beta_{i,j-1}) d_{S_{ij}}(x, y) = \sum_{i=1}^k |f(x)_i - f(y)_i| = \|f(x) - f(y)\|_1 = d(x, y),$$

so we have written d as a sum of cut semimetrics. □

Theorem 1.6. Assume (V, d_{min}) embeds into L_1 with distortion at most K, then $K^{-1} \varphi^*(C, D) \leq \varphi(C, D) \leq \varphi^*(C, D)$.

Proof. Let $f \colon (V_1, d_{min}) \to L_1$ be an embedding with distortion at most K. Let $d(x, y) = \|f(x) - f(y)\|_1$. Since $\mathrm{dist}(f) \leq K$, there exists $a > 0$ such that $a d_{min}(x, y) \leq d(x, y) \leq K a d_{min}(x, y)$ for all $x, y \in V$. By lemma 1.5, there exists cuts $(S_i, V \setminus S_i)$, $1 \leq i \leq k$ and constants $\alpha_i \geq 0$, $i = 1, ..., k$ such that $d = \sum_{i=1}^k \alpha_i d_{S_i}$. Then

$$\varphi(C, D) = \frac{\sum_{uv \in E} C(uv) d_{min}(u, v)}{\sum_{u,v \in V} D(u, v) d_{min}(u, v)} \geq \frac{1}{K} \frac{\sum_{uv \in E} C(uv) d(u, v)}{\sum_{u,v \in V} D(u, v) d(u, v)} = \frac{1}{K} \frac{\sum_{i=1}^k \gamma_i}{\sum_{i=1}^k \delta_i},$$

where $\gamma_i = \alpha_i \sum_{uv} C(uv) d_{S_i}(u,v)$ and $\delta_i = \alpha_i \sum_{u,v \in V} D(uv) d_{S_i}(u,v)$. Let $I = \{i : \delta_i > 0\}$. The above becomes

$$\geq \frac{1}{K} \frac{\sum_{i \in I} (\gamma_i/\delta_i)\delta_i}{\sum_{i \in I} \delta_i} \geq \frac{1}{K} \min_{i \in I} \frac{\gamma_i}{\delta_i} \geq \frac{1}{K} \varphi^*(C, D).$$

□

Definition. Let $f \colon M \to N$ be a map between metric spaces. Assume there exists increasing functions $\rho_1, \rho_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$ ($s \leq t \implies \rho_1(s) \leq \rho_1(t)$) such that

$$\rho_1(d(x,y)) \leq d(f(x), f(y)) \leq \rho_2(d(x,y)) \qquad \forall x, y \in M. \tag{2}$$

We say f is a *coarse embedding* if in addition to (2), $\rho_1(t) \to \infty$ as $t \to \infty$.

Example. Let $f \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ by $f(x,t) = x$. This is a coarse embedding with $\rho_1(t) = \max(0, t-1)$ and $\rho_2(t) = t$.

Definition. We say f is a *uniform embedding* if in addition to (2), $\rho_2(t) \to 0$ as $t \to 0^+$ and $\rho_1(t) > 0$ for all $t > 0$. Equivalently this says f is uniformly continuous, injective; $f^{-1} \colon f(M) \to M$ is uniformly continuous.

Proposition 1.7. For all $1 < q < \infty$ there exists a map $T \colon L_1(\Omega, \mu) \to L_q(\Omega \times \mathbb{R}, \nu)$ which is simultaneously a uniform and coarse embedding. (Here $\nu = \mu \otimes \lambda$ is the product measure of μ and the Lebesgue measure λ.)

Proof. Define T as follows. For $f \in L_1(\Omega, \mu)$,

$$Tf(\omega, t) = \begin{cases} +1 & \text{if } 0 < t \leq f(\omega), \\ -1 & \text{if } f(\omega) \leq t \leq 0 \\ 0 & \text{else.} \end{cases}$$

Note that $Tf \in L_\infty(\Omega \times \mathbb{R})$. For $f, g \in L_1(\Omega, \mu)$,

$$|Tf(\omega,t) - Tg(\omega,t)| = \begin{cases} 1 & \text{if } g(\omega) \leq t \leq f(\omega), \\ 1 & \text{if } f(\omega) \leq t \leq g(\omega). \end{cases}$$

So

$$\int_\Omega \int_\mathbb{R} |Tf(\omega,t) - Tg(\omega,t)|^q \, dt \, d\mu(\omega) = \int_\Omega |f(\omega) - g(\omega)| \, d\mu(\omega) = \|f - g\|_1 .$$

So $\|Tf - Tg\|_q^q = \|f - g\|_1$. This shows that $Tf \in L_q(\Omega \times \mathbb{R})$.

If $\rho_1(t) = \rho_2(t) = t^{1/q}$, then $\rho_1(\|f-g\|_1) = \|Tf - Tg\|_q = \rho_2(\|f - g\|_1)$. And $\rho_1(t) \to \infty$ as $t \to \infty$, and $\rho_2(t) \to 0$ as $t \to 0^+$ and $\rho_1(t) > 0$ for all $t > 0$. □

Proposition 1.8. For $1 \leq p < q < \infty$ there exists $T \colon L_p(\Omega, \mu) \to L_q(\Omega \times \mathbb{R}, \nu; \mathbb{C}) = \{f \colon \Omega \times \mathbb{R} \to \mathbb{C} : f \text{ measurable}, \int_{\Omega \times \mathbb{R}} |f|^q < \infty\}$, which is simultaneously a coarse and a uniform embedding.

Lemma 1.9. For all $0 < \alpha < 2\beta$ there exists $c_{\alpha,\beta} > 0$ such that

$$f(x) := \int_\mathbb{R} \frac{(1 - \cos(tx))^\beta}{|t|^{\alpha+1}} \, dt = c_{\alpha,\beta} |x|^\alpha.$$

Proof. First check the integrand is in $L_1(\mathbb{R})$: as $t \to 0$, $(1-\cos(tx))^\beta \sim |t|^{2\beta}$, so the integrand $\sim |t|^{2\beta-\alpha-1}$, so is integrable on, say, $(-1,1)$, since $2\beta-\alpha-1 > -1$. As $|t| \to \infty$, $(1-\cos(tx))^\beta$ is bounded, so the integrand is $\sim |t|^{-\alpha-1}$, which is integrable on $\mathbb{R} \setminus (-1,1)$, since $-\alpha - 1 < -1$.

For $x > 0$,

$$f(x) = x^\alpha \int_{\mathbb{R}} \frac{(1-\cos(tx))^\beta}{|tx|^{\alpha+1}} x \, dt = x^\alpha \int_{\mathbb{R}} \frac{(1-\cos(s))^\beta}{|s|^{\alpha+1}} x \, dt = x^\alpha f(1).$$

Also, $f(0) = 0$, $f(-x) = f(x)$ for all x. So $f(x) = |x|^\alpha f(1)$ for all x. □

Proof of Proposition 1.8. [A possible attempt is $Tf(\omega,t) = \frac{(1-\cos(tf(\omega)))^{1/2}}{|t|^{(p+1)/q}}$. Then

$$\int_{\mathbb{R}} |Tf(\omega,t)|^q \, dt = \int_{\mathbb{R}} \frac{(1-\cos(tf(\omega)))^{q/2}}{|t|^{p+1}} \, dt = \|f(\omega)\|^p .$$

The problem is taking $Tf - Tg$. The clever thing is that T is exponential.]

Define

$$Tf(\omega,t) = \frac{1-e^{itf(\omega)}}{|t|^{(p+1)/q}}.$$

For $\theta \in \mathbb{R}$, $\left|1-e^{i\theta}\right| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2-2\cos\theta} = \sqrt{2}(1-\cos\theta)^{1/2}$. Then

$$\begin{aligned}
\|Tf\|_q^q &= \int_\Omega \int_{\mathbb{R}} \frac{2^{q/2}(1-\cos(tf(\omega)))^{q/2}}{|t|^{p+1}} \, dt \, d\mu(\omega) \\
&= \int_\Omega 2^{q/2} C_{p,q/2} |f(\omega)|^p \, d\mu(\omega) && \text{by Lemma 8, } \alpha = p, \beta = q/2 \\
&= 2^{q/2} C_{p,q/2} \|f\|_p^p .
\end{aligned}$$

Given $f, g \in L_p(\Omega)$,

$$\left| e^{itf(\omega} - e^{itg(\omega)} \right| = \left| 1 - e^{it(f(\omega)-g(\omega))} \right| .$$

Apply above computation with f replaced with $f-g$ to get

$$\|Tf - Tg\|_q^q = 2^{q/2} C_{p,q/2} \|f-g\|_p^p .$$

Take $\rho_1(t) = \rho_2(t) = \sqrt{2} C_{p,q/2}^{1/q} t^{p/q}$. □

Corollary 1.10. For $1 \le p < q < \infty$ there exists $T\colon L_p \to L_q$ which is a simultaneously coarse and uniform embedding.

Apply proposition 8 with $(\Omega,\mu) = ([0,1],\lambda)$ to get embedding $L_p \to L_q([0,1] \times \mathbb{R}; \mathbb{C})$. Then $L_q([0,1]\times\mathbb{R};\mathbb{C}) \hookrightarrow_2 L_q([-1,1]\times\mathbb{R})$ by $f \mapsto \tilde{f}$ where

$$\tilde{f}(s,t) = \begin{cases} \operatorname{Re} f(s,t) & s \in (0,1] \\ \operatorname{Im} f(-s,t) & s \in [-1,0). \end{cases}$$

Since $L_q([-1,1]\times\mathbb{R})$ is separable, it embeds isometrically into L_q.

Definition. Given families $(M_\alpha)_{\alpha\in A}$ of metric spaces, a family $f_\alpha\colon M_\alpha \to N_\alpha$ a family of coarse embeddings is *uniformly coarse* if there exists increasing $\rho_1, \rho_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho_1(t) \to \infty$ as $t \to \infty$ and

$$\rho_1(d(x, y)) \leq d(f(x), f(y)) \leq \rho_2(d(x, y)) \qquad \forall x, y \in M, \forall \alpha \in A.$$

There are many connections of metric embeddings with other fields of mathematics, for example in geometry. The following two statements are non-examinable.

Theorem (Yu)**.** If M is a uniformly discrete metric space (every element is separated by at least $\delta > 0$) with bounded geometry (the number of points in any radius R is bounded by some $B(R)$)and M coarsely embeds into Hilbert space, then the coarse Baum-Connes conjecture holds for M.

Theorem (Kaspanov,Yu)**.** Same M, if M coarsely embeds into a uniformly convex Banach space then the coarse geometric Novikov conjecture hods for M.

2 Fréchet embeddings, Aharoni's theorem

Theorem 2.1 (Fréchet embedding)**.** Any metric space M embeds isometrically into $\ell_\infty(M)$. If $|M| = n < \infty$ then it isometrically embeds into ℓ_∞^{n-1}. If M is separable, then it embeds isometrically into $\ell_\infty = \ell_\infty(\mathbb{N})$.

Proof. Fix $x_0 \in M$. Define $f\colon M \to \ell_\infty(M)$ by $f(x) = d(\cdot, x) - d(\cdot, x_0)$. Then for all $y \in M$, $|f(x)(y)| = |d(y, x) - d(y, x_0)| \le d(x, x_0)$. So $f(x) \in \ell_\infty(M)$. Observe that for every $x, z \in M$, $\|f(x) - f(z)\|_\infty = \|d(\cdot, x) - d(\cdot, z)\|_\infty \le d(x, z)$ by the triangle inequality. To get the lower bound, $\|f(x) - f(z)\|_\infty \ge |f(x)(z) - f(z)(z)| = d(x, z)$.

In fact we can isometrically embed M into $\ell_\infty(M\backslash\{x_0\})$. If $M = \{x_0, ..., x_{n-1}\}$, then $M \to \ell_\infty^{n-1}$, $x \mapsto d(\cdot, x)$ works.

If M is separable, take a countable dense $S \subset M$. Then S embeds isometrically into ℓ_∞. This extends to an isometric embedding $M \to \ell_\infty$ (given $x \in M$, there exists $x_n \in S$ $x_n \to x$. Let $f(x) = \lim f(x_n)$. Since $f(x_n)$ Cauchy this limit exists. Check that this definition is independent of the choice of sequence).

Another proof: Let $f\colon M \to \ell_\infty(M)$ be an isometric embedding. Then $X = \overline{\text{span}} f(M)$ is a separable Banach space. By Proposition 1.2, $X \hookrightarrow_1 \ell_\infty$. $\square$

Definition. Let $m_\infty(n)$ be the least m such that every n-element metric space embeds isometrically into ℓ_∞^m. By Theorem 2.1, $m_\infty(n) \le n - 1$ for all $n \in \mathbb{N}$.

Aim. There exists $c > 0$, $m_\infty(n) \ge n - cn^{2/3} \log n$ for all $n \ge 2$ (due to K Ball).

Background.

(i) Ramsey Theory: $\forall t \in \mathbb{N}\ \exists n \in \mathbb{N}$ if edges of K_n are red-blue coloured, then there exists a monochormatic copy of K_t in K_n. Let $R(t)$ be the least n that works. It is easy to see that $R(t) \le 4^t$. It is also known that $R(t) \ge c^t$ for some $c > 1$. Given graphs H_1, H_2, let $R(H_1, H_2)$ be the least n s.t. whenever edges of K_n are red-blue coloured, either there exists a red copy of H_1 or there exists a blue copy of H_2 inside of K_n. So $R(t) = R(K_t, K_t)$. We can see that this exists. If $t = \max\{|H_1|, |H_2|\}$ (the *order* $|G|$ of a graph is the number of vertices), then $R(H_1, H_2) \le R(t)$.

(ii) A graph $G = (V, E)$ is *bipartite* if there exists a partition $V = V_1 \cup V_2$ s.t. $\forall x, y \in V$, $xy \in E \implies x \in V_1, y \in V_2$ or $x \in V_2, y \in V_1$. The vertices $V_{1,2}$ are called *vertex classes*. If $E = \{xy : x \in V_1, y \in V_2\}$, then G is the *complete bipartite graph.* This is denoted K_{V_1,V_2}. Denote $K_{m,n}$ = any K_{V_1,V_2} with $|V_1| = m, |V_2| = n$. Observe $K_{2,2} = C_4$.

(iii) Given a graph G, its complement $\bar{G}$ has vertex set $V(\bar{G}) = V(G)$, and $E(\bar{G})$ is the complement of $E(G)$, i.e. $xy \in E(\bar{G}) \iff xy \notin E(G)$.

Definition. For a graph G, define a metric ρ:

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } xy \in E \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 2.2. If (G, ρ) embeds isometrically into ℓ_∞^k, then edges of $\bar{G}$ can be covered by $\le k$ complete bipartite subgraphs of $\bar{G}$.

Proof. Let $f\colon (G,\rho) \to \ell_\infty^k$ be isometric. Let $\alpha_i = \max_{x\in G} f(x)_i$, $\beta_i = \min_{x \in G} f(x)_i$, $i = 1, ..., k$. Observe $\alpha_i - \beta_i = \max x, y \in G(f(x)_i - f(y)_i) \leq \max_{x,y\in G} \|f(x) - f(y)\|_\infty \leq$ 2. Let $I = \{i = 1, ..., k : \alpha_i - \beta_i = 2\}$. Then $xy \in E(\bar{G}) \iff \exists i \in I |f(x)_i - f(y)_i| = 2 \iff \exists i \in I f(x)_i = \alpha_i, f(y)_i = \beta_i$ or vice versa.

Let $V_{i1} = \{x : f(x)_i = \alpha_i\}$ and $V_{i2} = \{x : f(x)_i = \beta_i\}$. Then $E(\bar{G}) = \bigcup_{i\in I} E(K_{V_{i1},V_{i2}})$, and $|I| \leq k$. □

Theorem 2.3. There exists $C > 0$, $\forall n \geq 2$, $m_\infty(n) \geq n - Cn^{2/3} \log n$.

Proof. We will use the following result: $\exists \alpha > 0$, $R(C_4, K_t) > \alpha(t/\log t)^{3/2}$ (Spencer uses probabilistic method). Now there exists $b > 0, \forall n$ if $t = \lceil bn^{2/3} \log n \rceil$, then $n < \alpha(t/\log t)^{3/2} < R(C_4, K_t)$. [Roughly: $n = (t/\log t)^{3/2} \implies t = n^{2/3} \log t$, so $\log t = 2/3 \log n + \log\log t$, $\log t \sim \log t - \log\log t \sim \log n$. So $t \sim n^{2/3} \log n$.] Fix $n \in \mathbb{N}$, let $t = \lceil bn^{2/3} \log n \rceil$. So $n < R(C_4, K_t)$, so there exists a red-blue colouring of K_n without red C_4 or blue K_t. Let G be the blue graph. Let $k = m_\infty(n)$. Since (G, ρ) embeds isometrically into ℓ_∞^k, by Lemma 2.2, $\bar{G}$ = red graph is covered by $\leq k$ complete bipartite subgraphs. Since $C_4 = K_{2,2} \not\subset \bar{G}$, one vertex class in each complete bipartite subgraph is of size 1. So there exists $\leq k$ vertices s.t. every edge in $\bar{G}$ is adjacent to one of them. Since $K_t \not\subset G$, it follows that $n \leq k+t-1$, so $k = m_\infty(n) \geq n-t+1 \geq n-Cn^{2/3} \log n$ for some absolute constant C. □

Remark. Since $R(t) > C^t$ for some $C > 1$, this method won't give better than $n - C \log n$ lower bound on $m_\infty(n)$.

Aim. $n - m_\infty(n) \to \infty$ as $n \to \infty$. (Pretrov, Solyanov(?), Zatitskivy(?))

Lemma 2.4 (Non-linear Hahn-Banach)**.** Let M be a metric space, $A \subset M$, $f\colon A \to \mathbb{R}$ a Lipschitz map with constant L. Then there exists a Lipschitz extension $\tilde{f}\colon M \to \mathbb{R}$ of f with constant L.

Proof. Fix $x_0 \in M \setminus A$. Define $\tilde{f}\colon A \cup \{x_0\} \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ \alpha & x = x_0. \end{cases}$$

We need to choose the right $\alpha \in \mathbb{R}$. Want

$$|\alpha - f(x)| \leq Ld(x_0, x) \qquad \forall x \in A,$$

i.e.,

$$f(y) - Ld(y, x_0) \leq \alpha \leq f(x) + Ld(x, x_0) \qquad \forall x, y \in A.$$

Such α exists if

$$f(y) - Ld(y, x_0) \leq f(x) + Ld(x, x_0) \qquad \forall x, y \in A \qquad (*).$$

Indeed, then take

$$\alpha = \sup_{y\in A}\{f(y) - Ld(y, x_0)\}.$$

To see (*),

$$f(y) - f(x) \leq Ld(x, y) \leq Ld(x, x_0) + Ld(x_0, y) \qquad \forall x, y \in A.$$

If $M \setminus A$ is finite or countable, then apply above recursively to get an extension. In general, use Zorn's Lemma to get a maximal extension $(\tilde{M}, \tilde{f})$. By above, $\tilde{M} = M$. □

Proposition 2.5. If A is a subset of a finite metric space M, and there exists an isometric embedding $f\colon A \to \ell_\infty^{|A|-k}$, then there exists an isometric embedding $g\colon M \to \ell_\infty^{|M|-k}$.

Proof. Let $f_i(x) = f(x)_i$, $1 \le i \le |A| - k$. Then each f_i is 1-Lipschitz so by Lemma 2.4, there exists a 1-Lipschitz extension $g_i\colon M \to \mathbb{R}$. Enumerate $M \setminus A$ as y_i, $|A| - k + 1 \le i \le |M| - k$ and let $g_i(x) = d(x, y_i)$, $x \in M$. Then $g\colon M \to \ell_\infty^{|M|-k}$, $g(x) = (g_i(x))_{i=1}^{|M|-k}$ is an isometric embedding. □

Background.

(i) Some more Ramsey Theory: For $s \ge 2$, $n \in \mathbb{N}$, $K_n^{(s)} = \{A \subset [n] : |A| = s\}$, $[n] = \{1, ..., n\}$. e.g. $K_n^{(2)} = K_n$. Then Ramsey says $\forall s, \forall t, \exists n$ if $K_n^{(s)}$ is red-blue coloured, then there exists a monochromatic copy of $K_t^{(s)}$, i.e. $\exists A \subset [n]$, $|A| = t$ s.t. $A^{(s)} = \{B \subset A : |B| = s\}$ is monochromatic.

Also $\forall s, \forall t, \forall c, \exists n$ if $K_n^{(s)}$ is c-coloured then $\exists$ monochromatic copy of $K_t^{(s)}$.

(ii) Recall that a *tree* T is a connected, acyclic graph. Equivalently, $\forall x, y \in T$, $\exists$ unique path x to y. If $\mathrm{diam}(T) = \max x, y \in T d(x, y) \le 4$, then there exists $c \in T$ $\forall x$ $d(c, x) \le 2$. Call this c a *centre* of T. Vertices in $\Gamma(c) = \{a \in T : ac \in E\}$ are the *main vertices*. Every other vertex is connected to a unique main vertex.

(iii) *Oriented graph.* An *orientation* of a graph G is an assignment of direction for each edge: if $e = xy \in E(S)$, there are two choices $\overrightarrow{xy}$ or $\overrightarrow{yx}$. This is called *alternating* if $\forall x$ either $\forall y \in \Gamma(x)$ we have $\overrightarrow{xy}$ (x is a *source*) or $\forall y \in \Gamma(x)$ we have $\overrightarrow{yx}$ (x is a *sink*). [The name comes from an alternating path, because once we make a choice on one edge, all the other edges are alternating in direction.] A connected graph has 0 or 2 alternating orientations. It has 0 iff it has an odd cycle, i.e. not bipartite. A tree has exactly two alternating orientations.

(iv) A metric space $\{x_1, ..., x_n\}$ is *generic* if the $\binom{n}{2}$ distances $d(x_i, x_j)$, $1 \le i < j \le n$ are linearly independent over $\mathbb{Q}$.

Theorem 2.6. For every $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ for all $n \ge N$, $m_\infty(n) \le n - k$.

Proof. Step 1. We can restrict to generic metric spaces. *Proof.* Let $M = \{x_1, ..., x_n\}$ with metric d be an arbitrary metric space. For $j \in \mathbb{N}$, we can pick $\alpha_{rs} \in (\frac{1}{2j}, \frac{1}{j})$, $1 \le r < s \le n$ s.t. $d_j(x_r, x_s) = d(x_r, x_s) + \alpha_{rs}$ defines a generic metric.

Suppose $\forall j, \exists$ isometric embedding $f_j\colon (\{x_1, ..., x_n\}, d_j) \to \ell_\infty^m$ for some m. WLOG $\mathrm{im}(f_j)$ is bounded independent of j. By compactness, after passing to a subsequence, we have $f(x_r) = \lim_{j\to\infty} f_j(x_r)$ exists $\forall r$. Then $f\colon (M, d) \to \ell_\infty^m$ is an isometric embedding.

From now on, M is an n-element generic metric space and the elements of M are real numbers.

Step 2. Assume $f\colon M \to \mathbb{R}$ is 1-Lipschitz. We define a graph $G(f)$ with vertex set M and $xy \in E \iff |f(x) - f(y)| = d(x, y)$. We will orient an edge xy s.t. for $\overrightarrow{xy}$ we have $f(x) - f(y) = d(x, y)$.

Example. $f(x) = d(x, a)$, then this is a star with centre a and every edge pointing to it. For $x \neq y$ in $M \setminus \{a\}$, $f(x) - f(y) < d(x, y)$.

We have functions $f_1, ..., f_m \colon M \to \mathbb{R}$, $f \colon M \to \ell_\infty^m$ given by $f(x) = (f_i(x))_{i=1}^m$. Then f is an isometric embedding $\iff$ the f_i are 1-Lipschitz, $\forall x \neq y, \exists i, xy \in E(G(f_i))$. So M embeds isometrically into ℓ_∞^m $\iff$ the edges of the complete graph on M can be covered by at most m *Lipschitz graphs.*

Step 3. Let T be a tree on M with $\operatorname{diam}(T) \leq 4$. For fixed $x_0 \in T$, $\alpha \in \mathbb{R}$, alternating orientation of T, consider the unique function $f \colon M \to \mathbb{R}$ where $f(x_0) = \alpha$, $f(x) - f(y) = d(x, y)$ for all $\overrightarrow{xy} \in E(T)$. Then f is 1-Lipschitz $\iff$ for every path $wxyz$ in T, $d(w, x) + d(y, z) < d(x, y) + d(w, z)$. [We only need $\Leftarrow$ direction.]

Proof. Given $x, y \in T$, we need $|f(x) - f(y)| \leq d(x, y)$. If $d_T(x, y) = 0$ or 1, then this is true [d_T is the graph distance on T.]

If we have a path xzy, then $|f(x) - f(y)| = |f(x) - f(z) + f(z) - f(y)| = |d(x, z) - d(z, y)| < d(x, y)$ [here we use orientation.]

If we have a path $xwzy$ then $|f(x) - f(y)| = |f(x) - f(w) + f(w) - f(z) + f(z) - f(y)| = |d(x, w) - d(w, z) + d(z, y)|$. If this $= d(x, w) - d(w, z) + d(z, y)$ then $< d(x, y)$ by assumption. If this $= -d(x, w) + d(w, z) - d(z, y)$ then by the triangle inequality this is $< d(x, z) - d(z, y) < d(x, y)$.

If we have a path $xuwzy$ then $|f(x) - f(y)| = |d(x, u) - d(u, w) + d(w, z) - d(z, y)|$. WLOG this $= d(x, y) - d(u, w) + d(w, z) - d(z, y)$ because we have an even number of terms. By the assumption, this $< d(x, z) - d(z, y)$ and by the triangle inequality, this $< d(x, y)$. Thus we have proved step 3.

A tree T on M is *admissible* if it has diam ≤ 4 and satisfies the assumption in step 3.

Step 4. Given distinct points $c, a_1, ..., a_m$ in M, there exists a unique admissible tree on M with centre c and main vertices $a_1, ..., a_m$. Denote this by $T(c; a_1, ..., a_m)$.

Proof. Given $x \in M \setminus \{c, a_1, ..., a_m\}$, x can be joined to main vertex a $\iff$ for every main vertex $b \neq a$ we have $d(x, a) + d(c, b) < d(a, c) + d(x, b)$, i.e.

$$d(x, a) - d(a, c) < d(x, b) - d(c, b).$$

So a is uniquely determined.

Step 5. We colour $M^{(4)}$ using as colours elements of S_3 as follows: given $w < x < y < z$ in M, let $R_1 = d(w, x) + d(y, z)$, $R_2 = d(w, y) + d(x, z)$, $R_3 = d(w, z) + d(x, y)$. We give $wxyz$ colour ijk if $R_i > R_j > R_k$. This is a 6-colouring of $M^{(4)}$.

Main Claim. $\forall k \in \mathbb{N}, \forall c \in S_3, \exists t_c \in \mathbb{N}$, every (any) monochromatic metric space of size t_c and colour c can be covered by $\leq t_c - k$ admissible trees.

From main claim, let $t = \max_{c \in S_3} t_c$. By Ramsey $\exists N$ s.t. if $K_N^{(4)}$ is 6-coloured, then there exists a monochromatic $K_t^{(4)}$. So given $n \geq N$, an n-element metric space M, there exists a colour $c \in S_3$ and $A \subset M$, $|A| = t_c$ s.t. A is monochromatic. By main claim, the complete graph on A can be covered by $|A| - k$ admissible trees, so by step 2, A embeds isometrically into $\ell_\infty^{|A|-k}$. By Proposition 2.5, M embeds isometrically into ℓ_∞^{M-k}. So done. It remains to check the main claim.

Recall the *Main Claim*: $\forall c \in S_3, \forall k, \exists t$ every metric space M with $|M| = t$ and colour c can be coloured by $t - k$ admissible trees [edges of the complete graph of M].

Proof of Main Claim:

Case 1, $c = 213$. Then there does not exist M of colour c of size ≥ 5 ($t = 5$ will do). To see this, assume the contrary and aim for a contradiction. Fix $u < w < x < y < z$ in M. Then

$$d(u,w) + d(x,y) > d(u,y) + d(w,x);$$
$$d(w,x) + d(y,z) > d(w,z) + d(x,y);$$
$$d(u,y) + d(w,z) > d(u,w) + d(y,z).$$

Adding these gives $0 > 0$, a contradiction.

Case 2, $c = 312$. Same; just replace $>$ with $<$.

Case 3, $c = 132$. *Mini claim:* Assume if $|M| = n$, colour 132, then all but m edges of K_M can be covered by s admissible trees. Then if $|M| = n+2$ of colour 132, then all but $m-1$ edges of K_M can be covered by $s+2$ admissible trees.

Proof of mini claim. $|M| = n$, colour 132 and we have s trees that cover all but m edges. Let $ab, a < b$ be one of these edges. Let $|M'| = n+2$, colour 132. WLOG $M' = M \cup \{a', b'\}$, where $a < a' < b' < b$ and $M \cap ((a, a'] \cup [b', b)) = \emptyset$. Extend the s trees to the whole of M'. By step 4, add $T(a; a', b), T(b; a', b')$. Every $x \in M' \setminus \{a, a', b\}$ is joined to a' in $T(a; a', b)$ Every $x \in M' \setminus \{b, a', b'\}$ is joined to b' in $T(b; a', b')$. □

Apply mini claim: start with $|M| = k$ and $s = 0$, $m = \binom{k}{2}$. Apply mini claim m times to get M' with $t = |M'| = k + 2\binom{k}{2} = k^2$, $s = 2\binom{k}{2} = t - k$, $m = 0$.

Case 4, $c = 123$. We prove Main Claim by induction on k. For $k = 1$, $t = 1$ will do. I have 0 edges so 0 trees will do. Let $k \geq 1$ and assume t works for k. For $k+1$, we prove that $2t+3$ works. Take $M = \{-1, 0, 1, 2, ..., t+1, t+2, ..., 2t+1\}$. Consider $T(0; -1, 2), T(1; 0, 2), T(t+1+i; i, i+1), 1 \leq i \leq t$. This covers all edges except perhaps edges between vertices in $\{t+2, ..., 2t+1\}$. These can be covered by $t - k$ trees by the induction hypothesis. So we need $2 + t + t - k = 2t + 2 - k = |M| - (k+1)$.

Case 5, $c = 231$. We show $t = 2k$ works for k. Take $M = \{-k, ..., -1, 1, ..., k\}$, take trees $T(-i; -k, -k+1, ..., -i-1, 1, ..., k)$, $1 \leq i \leq k$. This works [a bit fiddly and uninteresting].

Case 6, $c = 321$. $t = 4k+1$ works. $M = \{0, 1, ..., 4k\}$, take trees $T(0; i, 4k+1-i)$, $1 \leq i \leq 2k$, $T(i; 2k+i, 2k+i+1, ..., 4k+1, i)$ $1 \leq i \leq k$. So the number of tree is $3k = |M| - (k+1)$.

□

Remark. $m_\infty(n)$ = least m s.t. every n-element subset of some $L_\infty(\Omega, \mu)$ embeds isometrically into ℓ_∞^m.

We define for $1 \leq p < \infty$, $m_p(n)$ = least m s.t. every n-element subset of some $L_p(\Omega, \mu)$ embeds isometrically into ℓ_p^m.

Remark. Note $m_1(n) \leq n!$ (Proposition 1.4), $m_2(n) = n-1$ (Example Sheet).

Theorem 2.7. For all $1 \leq p < \infty$ and for all $n \geq 2$, $m_p(n) \leq \binom{n}{2}$.

Remark. For $1 \leq p < 2$, this is essentially best possible. [Example sheet: $m_p(2n+1) \geq n$.]

Lemma 2.8 (Carathéodory's Theorem)**.** Given $L \subset \mathbb{R}^N$, then $\operatorname{conv} L = \{\sum_{i=0}^N t_i x_i : x_i \in L, t_i \geq 0, \forall i, \sum_{i=0}^N t_i = 1\}$. It follows that $\overline{\operatorname{conv}} L = \operatorname{conv} L$ if L is compact.

Proof. Given $x \in \operatorname{conv} L$, we write $x = \sum_{i=1}^m t_i x_i$. WLOG $m > N+1, t_i > 0, \forall i$. Then $x_1, ..., x_m$ are affinely dependent – this means $x_1 - x_2, x_1 - x_3, ..., x_1 - x_m$ are linearly dependent. There exists $\lambda_1, ..., \lambda_m$ not all zero with $\sum \lambda_i = 0$, $\sum \lambda_i x_i = 0$. For any $s \in \mathbb{R}$, $\sum (t_i - s\lambda_i) = 1$, $\sum (t_i - s\lambda_i) x_i = x$. For $s > 0$, $t_i - s\lambda_i \geq 0$ if $\lambda_i \leq 0$. So we take $s = \min\{t_i/\lambda_i : \lambda_i > 0\}$ $(\exists i, \lambda_i > 0)$. Now $t_i - s\lambda_i \geq 0, \forall i$ and $\exists i, t_i - s\lambda_i = 0$. □

Proof of Theorem 2.7. Fix $n \geq 2$. Given an n-tuple $M = (x_1, ..., x_n)$ in some $L_p(\Omega, \mu)$, let $\theta_M = (\|x_i - x_j\|_p^p)_{1 \leq i < j \leq n} \in \mathbb{R}^N$ where $N = \binom{n}{2}$. Let $C = \{\theta_M : M$ is an n-tuple in some $L_p(\Omega, \mu)\}$.

C is a cone: $\theta \in C, t > 0 \implies t\theta \in C$. Suppose $M = (x_1, ..., x_n)$ is an n-tuple in $L_p(\Omega, \mu)$, $M' = (y_1, ..., y_n)$ in $L_p(\Omega', \mu')$. Then consider $z_i = (x_i, y_i) \in L_p(\Omega \amalg \Omega')$. Then

$$\|z_i - z_j\|_p^p = (\theta_M)_{ij} + (\theta_{M'})_{ij} \qquad \forall 1 \leq i < j \leq n.$$

So $\theta_M + \theta_{M'} \in C$.

Let

$$K = C \cap \left\{ \theta \in \mathbb{R}^N : \sum_{1 \leq i < j \leq n} \theta_{ij} = 1 \right\}.$$

Say $\theta \in C$ is *linear* if there exists $(t_1, ..., t_n) \in \mathbb{R}^n$ s.t. $\theta_{ij} = |t_i - t_j|^p$. Let

$$\begin{aligned} L &= \{\theta \in K : \theta \text{ is linear}\} \\ &= \left\{ (|t_i - t_j|^p)_{1 \leq i < j \leq n} : t_1, ..., t_n \in \mathbb{R}, \sum_{1 \leq i < j \leq n} |t_i - t_j|^p = 1. \right\} \end{aligned}$$

Note L is compact. K is convex so $\operatorname{conv} L \subset K$.

Given $\theta \in K$, say $\theta = (\|x_i - x_j\|_p^p)_{1 \leq i < j \leq n}$, where $x_1, ..., x_n \in L_p(\Omega, \mu)$. Can approximate x_i with simple function y_i s.t. $\varphi = (\|y_i - y_j\|_p^p) \in K$. So we have a measurable partition $\Omega = \bigcup_{r=1}^R A_r$ of Ω s.t. $y_i|_{A_r}$ is constant $\forall i, r$. Let $\varphi_r = (\|y_i|_{A_r} - y_j|_{A_r}\|_p^p)_{1 \leq i < j \leq n}$. Then φ_r is linear and $\varphi = \sum_{r=1}^R \varphi_r$. Let $\alpha_r = \sum_{1 \leq i < j \leq n} (\varphi_r)_{ij}$. Then $\sum_{r=1}^R \alpha_r = 1$. So $\varphi = \sum_{r=1}^R \alpha_r (\varphi_r / \alpha_r) \in \operatorname{conv} L$. This shows $K \subset \overline{\operatorname{conv}} L$. By Lemma 2.8, $K = \operatorname{conv} L$, and every $\theta \in C$ is a sum $\theta = \sum_{r=1}^N \theta_r$, where θ_r is linear for all r. Note $\{\theta : \sum \theta_{ij} = 1\}$ is $(N-1)$-dimensional. For each r, there exists $t_{ri} \in \mathbb{R}$ with $\theta_r = (|t_{r,i} - t_{r,j}|^p)_{1 \leq i < j \leq n}$. If $\theta = \theta_M$, $M = (x_1, ..., x_n) \in L_p(\Omega, \mu)^n$, define $f \colon M \to \ell_p^N$ by using these as coordinates: $f(x_i) = (t_{r,i})_{r=1}^R$. Then one line to check that this works. For $1 \le i < j \le n$,

$$\|f(x_i) - f(x_j)\|_p^p = \sum_r |t_{r,i} - t_{r,j}|^p = \sum_r (\theta_r)_{ij} = \theta_{ij} = \|x_i - x_j\|_p^p .$$

□

Theorem 2.9 (Aharoni's Theorem)**.** For any $\epsilon > 0$, any separable metric space embeds into c_0 with distortion $\le 3 + \epsilon$.

Motivation. Given Banach spaces X, Y, if X bilipschitzly embeds into Y, must X be isomorphically embed into Y? Yes, if Y is separable and there exists a Banach Space W such that $Y \sim W^*$. Theorem 9 shows that in general, the answer is no.

Notation. (i) In a metric space M, for $x \in M$ and $\delta > 0$ let $B_\delta(x) = \{y \in M : d(y,x) \le \delta\}$. $A \subset M$ is *δ-dense* in M if $\forall x \in M, d(x, A) < \delta$.

(ii) Given a set S, let $c_0(S) = \{f \in \ell_\infty(S) : \forall \epsilon > 0 \{s \in S : |f(s)| > \epsilon\}$ is finite$\}$. So $c_0 = c_0(\mathbb{N}) \cong c_0(S)$ for S countably infinite.

Idea. We will have a countable set S and a subset $M_S \subset M$ and we use maps $f \colon M \to c_0(S), f(x) = (d(x, M_S))_{s \in S}$. Fix $\delta > 1$, for $x \ne y$ in M, $\delta^n \le d(x,y) \le \delta^{n+1}$ for some $n \in \mathbb{Z}$. We will have $c \in M$ (a centre). One of x or y, say x, has $d(c,x) > \delta^n/2$. We will partition $M \setminus B_{\delta^n/2}(c)$.

Lemma 2.10. Let M be a separable metric space, $\lambda > 2, a > 0, N \subset M$. Then there exists subsets $M_1, M_2, \ldots$ of N such that

(i) $\forall x \in N, \exists i, d(x, M_i) < a$;

(ii) $\forall x \in M$, the set $\{i : d(x, M_i) < (\lambda - 1)a\}$ is finite;

(iii) $\forall i$, $\operatorname{diam}(M_i) \le 2\lambda a$.

Proof. WLOG $a = 1$ (just replace the distance d by d/a). M is separable, hence so is N, so there exists a countable sets

$$Z \subset N, \text{ which is 1-dense in } N,$$
$$Y \subset M, \text{ which is 1-dense in } M.$$

WLOG $Z \subset Y$ (replace Y by $Z \cup Y$). Enumerate Y as $y_1, y_2, y_3, \ldots$. Let $M_i = (B_\lambda(y_i) \cap Z) \setminus \bigcup_{j<i} M_j$. Then $\forall i, M_i \subset Z \subset N$, and $\forall i, M_i \subset B_\lambda(y_i)$. So $\operatorname{diam}(M_i) \le 2\lambda$. This shows (iii).

Given $x \in N$, there exists i such that $y_i \in Z$ and $d(x, y_i) < 1$. Then $y_i \in B_\lambda(y_i) \cap Z \subset \bigcup_{j=1}^i M_j$. So there exists $j \le i$ such that $d(x, M_j) < 1$. This shows (i).

Given $x \in M$, there exists i_0 such that $d(x, y_{i_0}) < 1$. If $d(x, M_i) < \lambda - 1$, then $d(y_{i_0}, M_i) < \lambda$ by the triangle inequality. For $i > i_0$, $y \in M_i$. Since $y_{i_0} \in \bigcup_{j \le i_0} M_j$ and $M_i \cap \bigcup_{j \le i_0} M_j = \emptyset$, we have $d(y_{i_0}, y) \ge \lambda$ so $d(y_{i_0}, M_i) \ge \lambda$. So $\{i : d(x, M_i) < \lambda - 1\}$ has at most i_0 elements. This shows (ii). □

Proof of Theorem 9, Assonad. Given separable metric space M and $\epsilon > 0$, choose $\lambda > 2$, $\eta > 0$ such that $\frac{3\lambda}{\lambda-2}(1+\eta) < 3+\epsilon$ [first choose λ so that $\frac{3\lambda}{\lambda-2} < 3+\epsilon$]. For $k \in \mathbb{Z}$, let $a_k = (1+\eta)^{-k}$. Fix a centre $c \in M$, let $M_k = M \setminus B_{3\lambda a_k/2}(c)$. Apply Lemma 10 to M, $N = M_k$, $a = a_k$ to get subsets $M_{k,1}, M_{k,2}, \ldots$ satisfying (i),(ii),(iii) in Lemma 10 with $M_{k,i}$ in place of M_i.

Let $S = \{(k,i) : k \in \mathbb{Z}, i = 1, 2, \ldots\}$. For $x \in M$, let $f_{k,i}(x) = [(\lambda-1)a_k - d(x, M_{k,i})] \vee 0$. Let $f(x) = (f_{k,i}(x))_{(k,i)\in S}$.

We first prove that $f(x) \in c_0(S)$. Since $(\lambda-1)a_k \to 0$ as $k \to \infty$, enough to show that for any $s \in \mathbb{Z}$, $T = \{(k,i) : f_{k,i}(x) \geq (\lambda-1)a_s\}$ is finite. For $k > s$, we have $f_{k,i}(x) \leq (\lambda-1)a_k < (\lambda-1)a_s$ so $(k,i) \notin T$ for all $k > s$ and for all i. For the other direction, since $a_k \to \infty$ as $k \to -\infty$, $\exists r < s$ s.t. $d(x,c) < (\frac{\lambda}{2}+1)a_r$. For $k < r$, $d(x,c) < (\frac{\lambda}{2}+1)a_k$, so $\forall i$,

$$d(x, M_{k,i}) \geq d(x, M \setminus B_{3\lambda a_k/2}(c)) \geq \frac{3\lambda a_k}{2} - d(x,y) > (\lambda-1)a_k,$$

so $\forall k < r$, $\forall i$, $f_{k,i}(x) = 0$, so $(k,i) \notin T$. Finally, by Lemma 10, for each k, $\{i : f_{k,i}(x) > 0\} = \{i : d(x, M_{k,i}) < (\lambda-1)a_k\}$ is finite. So $T \subset \bigcup_{k=1}^{s}\{i : f_{k,i}(x) > 0\}$ is finite.

Now we have $f\colon M \to c_0(S)$. This is clearly 1-Lipschitz. For the lower bound, fix $x \neq y$ in M and choose k such that

$$3\lambda a_k < d(x,y) \leq 3\lambda a_k(1+\eta).$$

By the triangle inequality, both x and y cannot belong to $B_{3\lambda a_k/2}(c)$, so WLOG $x \in M_k$. By Lemma 10(i), there exists i such that $d(x, M_{k,i}) < a_k$. So $f_{k,i}(x) \geq (\lambda-1)a_k - a_k = (\lambda-2)a_k$.

Pick $w \in M_{k,i}$, $d(x,w) < a_k$. For any $z \in M_{k,i}$ we have

$$d(y,z) \geq d(y,x) - d(x,w) - d(w,z) > 3\lambda a_k - a_k - \operatorname{diam}(M_{k,i}) \geq (\lambda-1)a_k,$$

because $\operatorname{diam}(M_{k,i}) \leq 2\lambda a_k$. So $d(y, M_{k,i}) \geq (\lambda-1)a_k$ and $f_{k,i}(y) = 0$. So

$$\begin{aligned}\|f(x) - f(y)\|_\infty &\geq |f_{k,i}(x) - f_{k,i}(y)| \\ &\geq (\lambda-2)a_k \\ &= \frac{3\lambda a_k(1+\eta)}{3\lambda(1+\eta)}(\lambda-2) \\ &> \frac{d(x,y)}{3+\epsilon}.\end{aligned}$$

□

Remark. Here we are embedding into $c_0^+(S) = \{f\colon S \to \mathbb{R}^+ : f \in c_0(S)\}$. Pelant showed that

$$\sup_M \inf_{f\colon M \to c_0^+} \operatorname{dist}(f) = 3,$$

where the supremum is over all separable metric space M and the infimum is over all bilipschitz embeddings f.

Kalton and Lancien showed that

$$\sup_M \inf_{f\colon M \to c_0} \operatorname{dist}(f) = 2.$$

3 Bourgain's Embedding Theorem

For metric spaces X, Y, let

$$c_Y(X) = \inf\{\operatorname{dist}(f) : f\colon X \to Y \text{ a bilipschitz embedding}\}.$$

If $Y = L_p$, we write $c_p(X) = c_{L_p}(X)$, the *L_p-distortion* of X. $c_2(X)$ is called the *Euclidean distortion* of X. By Proposition 1.1, $c_p(X) \le c_2(X)$ for any finite X.

Theorem 3.1 (Dvoretzky's Theorem)**.** $\forall n \in \mathbb{N}, \forall \epsilon > 0, \exists N = N(n, \epsilon)$, s.t. every Banach space Y with $\dim Y \ge N$ contains a $(1+\epsilon)$-isomorphic copy of ℓ_2^n.

Remark.

(i) $N \le \exp(Cn/\epsilon^2)$ for some absolute constant C.

(ii) $c_Y(X) \le c_2(X)$ for every finite metric space and every infinite dimensional Banach space Y.

Aim. $c_2(X) \le C \log |X|$ for every finite X (Bourgain's embedding theorem).

From now on we fix a metric space X with $|X| = n$. Let $\mathcal{P}_X$ be the set of all partitions of X [pairwise disjoint non-empty subsets of X whose union is X]. For $P \in \mathcal{P}_X$, the elements of P are called *clusters*. For $x \in X$, we let $P(x)$ be the unique cluster to which x belongs. A *stochastic decomposition of* X is a probability measure Ψ on $\mathcal{P}_X$. Given $\Delta > 0$, $\epsilon\colon X \to (0, 1]$, we say Ψ is an *(ϵ, Δ)-padded decomposition* if

(i) $\forall P \in \mathcal{P}_X$ if $\Psi(P) > 0$ then $\forall C \in P$, $\operatorname{diam}(C) < \Delta$ [clusters can't be too big];

(ii) $\forall x \in X, \Psi(d(x, X \setminus P(x)) \ge \epsilon(x)\Delta) \ge \frac{1}{2}$.

Write $\operatorname{supp}(\Psi) = \{P \in \mathcal{P}_X : \Psi(P) > 0\}$, the *support* of Ψ.

Lemma 3.2. Let Ψ be an (ϵ, Δ)-padded decomposition of X, and let $1 \le q < \infty$. Then there exists 1-Lipschitz map $f\colon X \to \ell_q$ s.t.

(i) $\|f(x)\|_q \le \Delta, \forall x \in X$ (technical condition);

(ii) $\|f(x) - f(y)\|_q \ge C\epsilon(x)d(x, y), \forall x, y$ such that $d(x, y) \in [\Delta, 2\Delta)$, where C is an absolute constant (I think $C = \frac{1}{16}$) (lower Lipschitz condition).

Definition. For Banach spaces $X_1, X_2, \ldots$, for $1 \le q < \infty$ define $\left(\bigoplus_{i\ge1} X_i\right)_q$ to be the space of sequences $(x_i)_{i\ge1}$ s.t. $\sum_{i\ge1} \|x_i\|^q < \infty$. This is a Banach space with norm

$$\|(x_i)\| = \left(\sum_{i\ge1} \|x_i\|^q\right)^{1/q}.$$

Can also define $\left(\bigoplus_{i\ge1} X_i\right)_\infty$; $\|(x_i)\| = \sup_{i\ge1} \|x_i\|$. This has subspace $\left(\bigoplus_{i\ge1} X_i\right)_{c_0}$ of sequences $(x_i)_{i\ge1}$ such that $\|x_i\| \to 0$.

If $X_i = \ell_q$ for all i, then $\left(\bigoplus_{i\ge1} X_i\right)_q \cong \ell_q$.

Proof of Lemma 2. Fix $P \in \operatorname{supp}(\Psi)$. Let $C_1, C_2, ..., C_{m(P)}$ be the clusters of P. Let $U_1, ..., U_{2^{m(P)}}$ be all possible unions of the C_j. Fix $1 \leq j \leq 2^{m(P)}$ and define $f_{P,j} \colon X \to \mathbb{R}$ by

$$f_{P,j}(x) = \begin{cases} d(x, X \setminus P(x)) \wedge \Delta & \text{if } x \in U_j; \\ 0 & \text{otherwise.} \end{cases}$$

[Here $\wedge$ denotes the minimum.] We have $f_{P,j}(x) \leq \Delta$ for all $x \in X$.

Fix $x, y \in X$. If $P(x) \neq P(y)$ then

$$0 \leq f_{P,j}(x), f_{P,j}(y) \leq d(x,y).$$

If $P(x) = P(y)$, then either $x, y \in U_j$, in which case

$$f_{P,j}(z) = d(z, X \setminus P(x)) \wedge \Delta, \qquad z = x, y,$$

or $x, y \notin U_j$ in which case $f_{P,j}(x) = f_{P,j}(y) = 0$. In all cases $|f_{P,j}(x) - f_{P,j}(y)| \leq d(x,y)$. So $f_{P,j}$ is 1-Lipschitz.

Do this for each j, and define $f_P \colon X \to \ell_q^{2^{m(P)}}$ by

$$f_P(x) = \left(2^{-m(P)/q} f_{P,j}(x)\right)_{j=1}^{2^{m(P)}}.$$

So for all x,

$$\|f_P(x)\|_q = \left(\sum_{j=1}^{2^{m(P)}} 2^{-m(P)} f_{P,j}(x)^q\right)^{1/q} \leq \Delta$$

and for all x, y,

$$\|f_P(x) - f_P(y)\|_q = \left(\sum_{j=1}^{2^{m(P)}} 2^{-m(P)} |f_{P,j}(x) - f_{P,j}(y)|^q\right)^{1/q} \leq d(x,y).$$

So f_P is 1-Lipschitz.

Finally define

$$f \colon X \to \left(\bigoplus_{P \in \operatorname{supp}(\Psi)} \ell_q^{2^{m(P)}}\right)_{\ell_q} \hookrightarrow_{\cong} \ell_q,$$

by

$$f(x) = \left(\Psi(P)^{1/q} f_P(x)\right)_{P \in \operatorname{supp}(\Psi)}.$$

For all $x \in X$,

$$\|f(x)\|_q = \left(\sum_P \Psi(P) \|f_P(x)\|^q\right)^{1/q} \leq \Delta.$$

Similarly, f is 1-Lipschitz.

Fix x, y such that $d(x, y) \in [\Delta, 2\Delta)$. Let

$$E = \{P \in \operatorname{supp}(\Psi) : d(x, X \setminus P(x)) \geq \epsilon(x)\Delta\}.$$

Fix $P \in E$. If $x \in U_j$ and $y \notin U_j$ then$|f_{P,j}(x) - f_{P,j}(y)| \geq d(x, X \setminus P(x)) \geq \epsilon(x)\Delta$.

For $\frac{1}{4}$ of values of j we have $x \in U_j$, $y \notin U_j$ (note $P(x) \neq P(y)$, since $\forall C \in P, \operatorname{diam}(C) < \Delta \leq d(x, y)$). So

$$\|f_P(x) - f_P(y)\|_q \geq \left(\sum_{j, x \in U_j, y \notin U_j} 2^{-m(P)} |f_{P,j}(x) - f_{P,j}(y)|^q \right)^{1/q} \geq \epsilon(x)\Delta 4^{-1/q}.$$

Finally,

$$\|f(x) - f(y)\| \geq \left(\sum_{P \in E} \Psi(P) \|f_P(x) - f_P(y)\|^q \right)^{1/q} \geq \epsilon(x)\Delta 4^{-1/q} \Psi(E),$$

and this is

$$\geq \frac{\epsilon(x)\Delta}{4^{1/q} 2} \geq \frac{\epsilon(x)}{4^{1/q} 4} d(x, y) \geq \frac{1}{16} \epsilon(x) d(x, y).$$

□

Definition. Define the set of *relevant scales* to be

$$S(X) = \{\ell \in \mathbb{Z} : \exists x, y \in X, d(x, y) \in [2^\ell, 2^{\ell+1})\},$$

and $R(X) = |S(X)|$.

Example. If X is a connected graph with the graph distance, then $R(X) \leq \lceil \log_2 n \rceil$.

Definition. A map $f\colon X \to Y$, given $K, \tau > 0$, is called a *scaled-τ embedding with deficiency K* if f is 1-Lipschitz and $d(f(x), f(y)) \geq K^{-1} d(x, y)$ for all x, y such that $d(x, y) \in [\tau, 2\tau)$.

Proposition 3.3. Given $K > 0$, $1 \leq q < \infty$, assume $\forall \ell \in S(X), \exists f_\ell\colon X \to \ell_q$ a scale-2^ℓ embedding with deficiency K. Then $C_q(X) \leq K R(X)^{1/q}$.

Proof. Define $f\colon X \to \left(\bigoplus_{\ell \in S(X)} \ell_q\right) \cong \ell_q$ by $f(x) = (f_\ell(x))_{\ell \in S(X)}$. For all x, y, $\|f(x) - f(y)\| = \left(\sum_{\ell \in S(X)} \|f_\ell(x) - f_\ell(y)\|^q\right)^{1/q} \leq R(X)^{1/q} d(x, y)$. So f is $R(X)^{1/q}$-Lipschitz. Given $x \neq y$, there exists $\ell \in S(X)$ s.t. $d(x, y) \in [2^\ell, 2^{\ell+1})$. Then

$$\|f(x) - f(y)\| \geq \|f_\ell(x) - f_\ell(y)\| \geq \frac{1}{K} d(x, y).$$

So $c_q(X) \leq \operatorname{dist}(f) \leq K R(X)^{1/q}$. □

Corollary 3.4. If $\forall \ell \in S(X)$ there exists an $(\epsilon, 2^\ell)$-padded decomposition of X with $\epsilon(x) \geq \frac{1}{K}$ for all x, then $c_q(X) \leq C K R(X)^{1/q}$ $(1 \leq q < \infty)$.

Proof. Lemma 2 + Proposition 3. □

Remark. Actually, $c_q(X) \leq CKR(X)^{1/2 \wedge 1/q}$, because $c_q(X) \leq c_2(X)$.

Theorem 3.5 (Existence of a decomposition)**.** For every $\ell \in \mathbb{Z}, \exists(\epsilon, 2^\ell)$-padded decomposition of X with

$$\epsilon(x) = \left[16 + 16 \log\left(\frac{|B_{2^\ell}(x)|}{|B_{2^{\ell-3}}(x)|}\right)\right]^{-1}.$$

Remark. Note $\epsilon(x) \geq C\frac{1}{\log n}$, so by Corollary 4, $c_2(X) \leq C(\log n)\sqrt{R(X)}$.

Proof of Theorem 5. Fix $\ell \in \mathbb{Z}$ and set $\Delta = 2^\ell$. Fix an ordering $<$ on X. Consider a pair (π, α) where $\pi \in S_n$ (the symmetry group of X) and $\alpha \in (\frac{1}{4}, \frac{1}{2})$ and π, α are chosen uniformly at random and independently. To (π, α) there corresponds an element $P \in \mathcal{P}_X$ with clusters

$$C_y = B_{\alpha\Delta}(y) \setminus \bigcup_{z:\pi(z)<\pi(y)} B_{\alpha\Delta}(z), \qquad y \in X.$$

We throw away the empty clusters. This gives a random partition, so we have a stochastic decomposition.

Now we check this gives us an (ϵ, Δ)-padded decomposition. Note that $\operatorname{diam}(C_y) \leq 2\alpha\Delta < \Delta$ for ally $y \in X$. Now fix $x \in X$, $t \leq \frac{\Delta}{8}$. Let B (B for Bad) be the event that $d(x, X \setminus P(x)) \leq t$, i.e. $B_t(x) \not\subset P(x)$. The aim is to show that $\mathbb{P}(B) \leq \frac{1}{2}$ for $t = \epsilon(x)\Delta$. Then we would be done.

Note that B occurs $\iff$ $B_t(x) \not\subset C_y$ for all y. Assume $y \in X$ and $B_t(x) \cap C_y \neq \emptyset$. Then $B_t(x) \cap B_{\alpha\Delta}(y) \neq \emptyset$. So $d(x, y) \leq \alpha\Delta + t < \frac{\Delta}{2} + \frac{\Delta}{8} < \Delta$ by the triangle inequality. So $y \in B_\Delta(x)$. Let $b = |B_\Delta(x)|$ and $y_1 (= x), y_2, ..., y_b$ be the elements of $B_\Delta(x)$ in order of increasing distance to x.

Let $y \in X$ such that this necessary condition holds: $d(x, y) \leq \alpha\Delta + t$ and $\pi(y)$ is minimal in $<$. So $B_t(x)$ is disjoint from $\bigcup_{z:\pi(z)<\pi(y)} C_z = \bigcup_{z:\pi(z)<\pi(y)} B_{\alpha\Delta}(z)$ (by minimality). So $B_t(x) \subset C_y \iff B_t(x) \subset B_{\alpha\Delta(y)}$.

Now if B happens, then $B_t(x) \not\subset B_{\alpha\Delta}(y)$ and hence

$$d(x, y) > \alpha\Delta - t \geq \frac{\Delta}{4} - \frac{\Delta}{8} = \frac{\Delta}{8}.$$

Let $a = |B_{\Delta/8}(x)|$. Then $B_{\Delta/8}(x) = \{y_1, ..., y_a\}$. So the y above is y_k for some k with $a < k \leq b$.

So we proved that $B \subset \bigcup_{k=a+1}^b E_k$ where E_k is the event that $d(x, y_k) \leq \alpha\Delta + t$ with $\pi(y_k)$ is $<$-minimal with this property, and $d(x, y_k) > \alpha\Delta - t$.

Let $I_k = [d(x, y_k) - t, d(x, y_k) + t)$. Then $E_k \implies \alpha\Delta \in I_k$.

So $\mathbb{P}(B) \leq \sum_{k=a+1}^b \mathbb{P}(E_k) = \sum_{k=a+1}^b \mathbb{P}(E_k | \alpha\Delta \in I_k)\mathbb{P}(\alpha\Delta \in I_k)$. If $\alpha\Delta \in I_k$ then $d(x, y_j) \leq d(x, y_k) \leq \alpha\Delta + t$ for $1 \leq j \leq k$.

If in addition E_k occurs, we must have $\pi(y_k) < \pi(y_j)$ for all $j < k$. So

$$\begin{aligned}
\mathbb{P}(B) &\leq \sum_{k=a+1}^b \mathbb{P}(\pi(y_k) < \pi(y_j), \forall j < k | \alpha\Delta \in I_k)\mathbb{P}(\alpha\Delta \in I_k) \\
&= \sum_{k=a+1}^b \mathbb{P}(\pi(y_k) < \pi(y_j), \forall j < k)\mathbb{P}(\alpha\Delta \in I_k) \qquad \text{by independence of } \alpha, \pi \\
&\leq \sum_{k=a+1}^b \frac{1}{k}\frac{8t}{\Delta} \leq \frac{8t}{\Delta} \log\frac{b}{a} \leq \frac{1}{2} \qquad \text{if } t = \epsilon(x)\Delta.
\end{aligned}$$

So we have our (ϵ, Δ)-padded decomposition as desired. $\square$

Notation. For functions a, b on a set S and values in $\mathbb{R}^+$, $a \lesssim b$ means $\exists$ absolute constant C such that $a(s) \leq Cb(s)$ for all $s \in S$.

Theorem 3.6 (Gluing Lemma)**.** Let $1 \leq q < \infty, K > 0$. Assume $\forall \ell \in \mathbb{Z}, \exists$ a scale-2^ℓ embedding $f_\ell \colon X \to \ell_q$ of deficiency K and with $\|f_\ell(x)\| \leq 2^\ell$ for all $x \in X$. Then $c_q(X) \lesssim K^{1-1/q}(\log n)^{1/q}$.

Let's see how the Gluing Lemma implies Bourgain's Embedding Theorem.

Corollary 3.7 (Bourgain's Embedding Theorem)**.** $c_2(X) \lesssim \log n$.

Proof. By Theorem 5, there exists $(\epsilon, 2^\ell)$-padded decomposition for $X, \forall \ell \mathbb{Z}$ where $\epsilon(x) \geq C \frac{1}{\log n}$. By Lemma 2, for all $\ell \in \mathbb{Z}$ $\exists$ scale-2^ℓ embedding $f_\ell \colon X \to \ell_2$ with deficiency $K \leq C \log n$ and $\|f_\ell(x)\| \leq 2^\ell$ for all $x \in X$. By Theorem 6, $c_2(X) \leq C(\log n)^{1-1/2}(\log n)^{1/2} = C \log n$. $\square$

Now we will prove the Gluing Lemma. But first we need some notation.

Notation. For $x, y \in X, \ell \in \mathbb{Z}$, let

$$\gamma_\ell(x, y) = \begin{cases} x & \text{if } |B_{2^\ell}(x)| \geq |B_{2^\ell}(y)| \\ y & \text{otherwise.} \end{cases}$$

To prove the Gluing Lemma, we need two further lemmas.

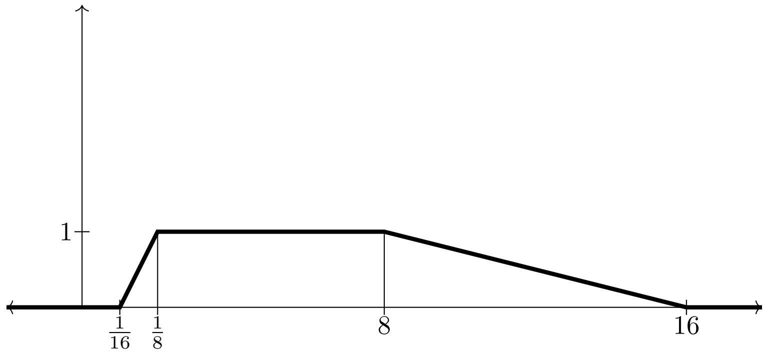
Lemma 3.8. Assume $\forall \ell \in \mathbb{Z}$ there exists 1-Lipschitz $h_\ell \colon X \to \ell_q$ $(1 \leq q < \infty)$ s.t. $\|h_\ell(x)\| \leq 2^\ell$ for all $x \in X$. Then there exists $H \colon X \to \ell_q$ s.t.

(i) $\mathrm{Lip}(H) \lesssim (\log n)^{1/q}$;

(ii) $\forall x, y \in X, \forall \ell \in \mathbb{Z}$ if $d(x, y) \in [2^\ell, 2^{\ell+1})$, then

$$\|H(x) - H(y)\| \geq \left(\log_2 \frac{|B_{2^{\ell+1}}(\gamma_{\ell-3}(x, y))|}{|B_{2^{\ell-3}}(\gamma_{\ell-3}(x, y))|} \right)^{1/q} \|h_\ell(x) - h_\ell(y)\|.$$

Proof. Let $\rho \colon \mathbb{R} \to \mathbb{R}^+$ be the function that is 0 on $(-\infty, \frac{1}{16}]$ then piecewise linear connecting $(\frac{1}{8}, 1)$, $(8, 1)$ and $(16, 0)$ and then 0 on $[16, +\infty)$. Note that $\mathrm{Lip}(\rho) \leq 16$.



Fix $t \in \{0, 1, 2, ..., \lceil \log_2 n \rceil - 1\}$. For $x \in X$ let

$$R(x, t) = \sup\{R : |B_R(x)| \leq 2^t\}.$$

This is 1-Lipschitz in x: given $x, y \in X$, if $|B_R(x)| \leq 2^t$, then $|B_{R-d(x,y)}(y)| \leq 2^t$ and so $R(y, t) \geq R - d(x, y)$. Take sup over R, $R(y, t) \geq R(x, t) - d(x, y)$.

Define

$$H_t \colon X \to \left(\bigoplus_{\ell \in \mathbb{Z}} \ell_q \right)_q \cong \ell_q$$

by

$$H_t(x) = \left(\rho\left(\frac{R(x, t)}{2^\ell} \right) h_\ell(x) \right)_{\ell \in \mathbb{Z}}.$$

Well-defined: Fix $x \in X$. Then $\rho\left(\frac{R(x,t)}{2^\ell}\right) = 0$ if $2^{\ell-4} \geq R(x, t)$ or $R(x, t) \geq 2^{\ell+4}$. Choose $m \in \mathbb{Z}$ s.t. $2^m \leq R(x, t) < 2^{m+1}$. Then $\rho\left(\frac{R(x,t)}{2^\ell}\right) = 0$ provided $2^{\ell-4} \geq 2^{m+1}$ or $2^m \geq 2^{\ell+4}$, so if $\ell \geq m + 5$ or $\ell \leq m - 4$. So $H_t(x)$ has ≤ 8 non-zero coordinates. So it is in ℓ_q.

Next we show H_t is Lipschitz with $\mathrm{Lip}(H_t) \leq 16 \times 17$. Note

$$\begin{aligned}
&\left\| \rho\left(\frac{R(x, t)}{2^\ell} \right) h_\ell(x) - \rho\left(\frac{R(y, t)}{2^\ell} \right) h_\ell(y) \right\| \\
&\leq \left| \rho\left(\frac{R(x, t)}{2^\ell} \right) - \rho\left(\frac{R(y, t)}{2^\ell} \right) \right| \|h_\ell(x)\| + \rho\left(\frac{R(y, t)}{2^\ell} \right) \|h_\ell(y) - h_\ell(x)\| \\
&\leq 16 \frac{1}{2^\ell} d(x, y) 2^\ell + d(x, y) \\
&= 17 d(x, y).
\end{aligned}$$

Since both $H_t(x), H_t(y)$ have ≤ 8 nonzero coordinates, we are done.

Now define

$$H \colon X \to \left(\bigoplus_{t=0}^{\lceil \log_2(n) \rceil - 1} \right)_q \cong \ell_q$$

by $H(x) = (H_t(x))_{t=0}^{\lceil \log_2 n \rceil - 1}$. It's clear that $\mathrm{Lip}(H) \lesssim (\log n)^{1/q}$. This proves (i).

To show (ii), fix $x, y \in X$, choose ℓ s.t. $d(x, y) \in [2^\ell, 2^{\ell+1})$. Then

$$\|H_t(x) - H_t(y)\| \geq \|h_\ell(x) - h_\ell(y)\| \qquad (*)$$

provided $\rho\left(\frac{R(x,t)}{2^\ell}\right) = \rho\left(\frac{R(y,t)}{2^\ell}\right) = 1$ which holds if $R(x, t), R(y, t) \in [2^{\ell-3}, 2^{\ell+3}]$.

This will follow if $|B_{2^{\ell-3}}(x)| \leq 2^t$, $|B_{2^{\ell+3}}(x)| > 2^t$ (same for y). So $(*)$ holds for all t such that

$$2^t \in [|B_{2^{\ell-3}}(x)|, |B_{2^{\ell+3}}(x)|) \cap [|B_{2^{\ell-3}}(y)|, |B_{2^{\ell+3}}(y)|).$$

WLOG $\gamma_{\ell-3}(x, y) = x$. Since $d(x, y) < 2^{\ell+1}$, $B_{2^{\ell+1}}(x) \subset B_{2^{\ell+3}}(y)$.

So $(*)$ holds if

$$2^t \in [|B_{2^{\ell-3}}(x)|, |B_{2^{\ell+1}}(x)|).$$

So

$$\|H(x) - H(y)\| = \left(\sum_t \|H_t(x) - H_t(y)\|\right)^{1/q}$$
$$\geq \left(\log_2 \frac{|B_{2^{\ell+1}}(x)|}{|B_{2^{\ell-3}}(x)|}\right)^{1/q} \|h_\ell(x) - h_\ell(y)\|.$$

□

Lemma 3.9. Let $1 \leq q < \infty$. Then there exists $H\colon X \to \ell_q$ such that

(i) $\mathrm{Lip}(H) \lesssim (\log n)^{1/q}$;

(ii) $\forall x, y \in X$, $\forall \ell \in \mathbb{Z}$, if $d(x,y) \in [2^\ell, 2^{\ell+1})$ and

$$\log_2 \left(\frac{|B_{2^{\ell-1}}(x)|}{|B_{2^{\ell-2}}(x)|}\right) < 1,$$

then $\|H(x) - H(y)\| \gtrsim d(x,y)$.

Proof. Fix $t \in \{1, 2, ..., \lceil \log_2 n \rceil\}$. Let W be a random subset of X where each $x \in X$ is placed in W independently at random with probability 2^{-t}. Let $\mathbb{P}_t$ be the resulting probability measure on $\mathcal{P}(X)$, the power set of X. So $\mathbb{P}_t(W) = 2^{-t|W|}(1-2^{-t})^{n-|W|}$ for any $W \subset X$. Note that $L_q(\mathcal{P}(X), \mathbb{P}_t) \cong \ell_q^{2^n}$ by

$$g \leftrightarrow \left(\mathbb{P}_t(W)^{1/q} g(W)\right)_{W \in \mathcal{P}(X)}.$$

Note

$$\|g\|_q^q = \int_{\mathcal{P}(X)} |g(W)|^q \, d\mathbb{P}_t(W)$$
$$= \sum_W \mathbb{P}_t(W) |g(W)|^q$$
$$= \left\| (\mathbb{P}_t(W)^{1/q} g(W))_W \right\|_q^q.$$

Define $H_t\colon X \to L_q(\mathcal{P}(X), \mathbb{P}_t) \cong \ell_q^{2^n}$ by $H_t(x) = (d(x, W))_W$. Then for all $x, y \in X$,

$$\|H_t(x) - H_t(y)\| = \left(\int_{\mathcal{P}(X)} |d(x,W) - d(y,W)|^q \, d\mathbb{P}_t(W)\right)^{1/q}$$
$$\leq d(x,y),$$

so H_t is 1-Lipschitz.

Define $H\colon X \to \left(\bigoplus_{t=1}^{\lceil \log_2 n \rceil} \ell_q^{2^n}\right)_q \hookrightarrow_{\cong} \ell_q$ by $H(x) = (H_t(x))_{t=1}^{\lceil \log_2 n \rceil}$. Then $\mathrm{Lip}(H) \lesssim (\log n)^{1/q}$. This shows (i).

To see (ii), fix $x, y \in X$, $\ell \in \mathbb{Z}$ such that $d(x,y) \in [2^\ell, 2^{\ell+1})$ and

$$\log_2 \left(\frac{|B_{2^{\ell-1}}(x)|}{|B_{2^{\ell-2}}(x)|}\right) < 1.$$

Fix $s \in \{1, 2, ..., \lceil \log_2 n \rceil\}$ such that $2^{s-1} \leq |B_{2^{\ell-1}}(x)| \leq 2^s$. Note $2^s \geq |B_{2^{\ell-2}}(x)| \geq 2^{s-2}$. Consider 4 events:

$$\begin{aligned} E_x &= \{W : d(x, W) \leq 2^{\ell-2}\} = \{W : W \cap B_{2^{\ell-2}}(x) \neq \emptyset\}, \\ F_x &= \{W : d(x, W) > 2^{\ell-1}\} = \{W : W \cap B_{2^{\ell-1}}(x) = \emptyset\}, \\ E_y &= \{W : d(y, W) \leq \frac{3}{2} 2^{\ell-2}\} = \{W : W \cap B_{\frac{3}{2}2^{\ell-2}}(y) \neq \emptyset\}, \\ F_y &= \mathcal{P}(X) \setminus E_y = \{W : W \cap B_{\frac{3}{2}2^{\ell-2}}(y) = \emptyset\}. \end{aligned}$$

Since $d(x, y) \geq 2^\ell$, $B_{2^{\ell-1}}(x) \cap B_{\frac{3}{2}2^{\ell-2}}(y) = \emptyset$, and hence any of E_x, F_x is independent of E_y, F_y.

Now we calculate the probabilities.

$$\begin{aligned} \mathbb{P}_s(E_x) &= 1 - (1 - 2^{-s})^{|B_{2^{\ell-2}}(x)|} \geq 1 - (1 - 2^{-s})^{2^{s-2}} \geq 1 - e^{-1/4} > 0, \\ \mathbb{P}_s(F_x) &= 1 - (1 - 2^{-s})^{|B_{2^{\ell-1}}(x)|} \geq 1 - (1 - 2^{-s})^{2^s} \geq (1 - \frac{1}{2})^2 = \frac{1}{4} > 0. \end{aligned}$$

So

$$\begin{aligned} \|H(x) - H(y)\| &\geq \|H_s(x) - H_s(y)\| \\ &= \left(\int_{\mathcal{P}(X)} |d(x, W) - d(y, W)|^q \, d\mathbb{P}_s(W) \right)^{1/q} \\ &\geq \left(\int_{E_x \cap F_y} + \int_{E_y \cap F_x} (\cdots) \right)^{1/q} \\ &\gtrsim (2^{(\ell-3)q} \mathbb{P}_s(F_y) + 2^{(\ell-3)q} \mathbb{P}_s(E_y))^{1/q} \\ &\gtrsim 2^{\ell+1} \geq d(x, y), \end{aligned}$$

as required. [Here we have used independence.] □

Proof of Theorem 6. Apply Lemma 8 with $h_\ell = f_\ell$ to get H, which we will call $F\colon X \to \ell_q$ such that $\mathrm{Lip}(F) \lesssim (\log n)^{1/q}$, and $\forall x, y \in X, \ell \in \mathbb{Z}$ if $d(x, y) \in [2^\ell, 2^{\ell+1})$, then

$$\|F(x) - F(y)\| \geq \left(\log_2 \frac{|B_{2^{\ell+1}}(\gamma_{\ell-3}(x, y))|}{|B_{2^{\ell-3}}(\gamma_{\ell-3}(x, y))|} \right)^{1/q} \|f_\ell(x) - f_\ell(y)\|.$$

Remember $\|f_\ell(x) - f_\ell(y)\| \geq \frac{1}{K} d(x, y)$.

From Theorem 5 and Lemma 2, we get $\forall \ell \in \mathbb{Z}$ a 1-Lipschitz $g_\ell\colon X \to \ell_q$ such that $\|g_\ell(x)\| \leq 2^\ell$ for all x and $\forall x, y \in X$, if $d(x, y) \in [2^\ell, 2^{\ell+1})$, then

$$\|g_\ell(x) - g_\ell(y)\| \gtrsim \left[16 + 16 \log \left(\frac{|B_{2^\ell}(x)|}{|B_{2^{\ell-3}}(x)|} \right) \right]^{-1} d(x, y).$$

Apply Lemma 8 with $h_\ell = g_\ell$ to get H which we call G here such that (i) and (ii) of Lemma 8 hold.

Let H be the function from Lemma 9. Define $\Phi\colon X \to (\ell_q \oplus \ell_q \oplus \ell_q)_q \cong \ell_q$ where $\Phi(x) = (F(x), G(x), H(x))$. Clearly we have $\mathrm{Lip}(\Phi) \lesssim (\log n)^{1/q}$. Fix

$x, y \in X$ with $d(x,y) \in [2^\ell, 2^{\ell+1})$ for some $\ell \in \mathbb{Z}$. Let $A = \log_2\left(\frac{|B_{2^{\ell+1}}(x)|}{|B_{2^{\ell-3}}(x)|}\right)$ and assume $\gamma_{\ell-3}(x,y) = x$. If $A < 1$ then by Lemma 9, $\|H(x) - H(y)\| \gtrsim d(x,y)$. If $A \geq 1$ then $\|F(x) - F(y)\| \geq A^{1/q}\frac{1}{K}d(x,y)$.

$$\|G(x) - G(y)\| \gtrsim \frac{A^{1/q}}{1+A}d(x,y).$$

Consider $A \geq K$ and $A \leq K$ to get $K^{-1+1/q}d(x,y)$ lower bound. So $\operatorname{dist}(\Phi) \lesssim K^{1-1/q}(\log n)^{1/q}$. □

4 Lower Bounds on Distortions, Poincaré Inequalities

In Section 3, we proved that $c_2(X) \lesssim \log|X|$ for any finite metric space X. Is this best possible? One might think that $c_2(X) \lesssim \sqrt{\log|X|}$.

Definition. For normed spaces X, Y we define the *Banach-Mazur distance*

$$d(X,Y) = \inf\{\|T\| \left\|T^{-1}\right\| : T\colon X \to Y \text{ is an onto isomorphism}\}.$$

[By convention $\inf \emptyset = \infty$.]

Note $1 \le \left\|T \circ T^{-1}\right\| \le \|T\| \left\|T^{-1}\right\|$, so $1 \le d(X,Y)$. Also $d(X,Z) \le d(X,Y) \times d(Y,Z)$ for all X, Y, Z. [If $T\colon X \to Y, S\colon Y \to Z$, then $\|ST\| \le \|S\| \|T\|$.] If $X \cong Y$ then $d(X,Y) = 1$. The converse is false in general.
Aside. Let $\mathcal{M}_n$ be the class of all n-dimensional normed spaces (we identify spaces that are isometrically isomorphic). On $\mathcal{M}_n$, $\log d$ is a metric and $\mathcal{M}_n$ is compact – the *Banach-Mazur compaction*.

Theorem (John's Lemma)**.** For any n-dimensional normed space X, $d(X, \ell_2^n) \le \sqrt{n}$.

Remark. (i) For all X, Y n-dimensional normed spaces, $d(X,Y) \le n$. [$\exists c > 0, \forall n, \operatorname{diam}(\mathcal{M}_n) \ge cn$ (Gluskin)].

(ii) For a general finite metric space X, the analogue of dimension, is $\log|X|$. This is to do with entropy. By analogy with John's Lemma, one might hope $c_2(X) \lesssim \sqrt{\log|X|}$.

Proof of John's Lemma. We can think of X as $\mathbb{R}^n$ with some norm $\|\cdot\|$. Let $K = B_X = \{x \in X : \|x\| \le 1\}$. This is a symmetric, convex body. [*Symmetric* means $\forall x \in K$, $-x \in K$, i.e. $K = -K$. *Body* means compact with nonempty interior.] Conversely, if K is a symmetric convex body, then $K = B_X$ where $X = (\mathbb{R}^n, \|\cdot\|)$ and $\|x\| = \inf\{t > 0 : x \in tK\}$. An *ellipsoid* is a subset $E \subset \mathbb{R}^n$ such that $E = T(B_{\ell_2^n})$ where $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection. Then $n^{-1/2}E \subset K \subset E \iff d(X, \ell_2^n) \le \sqrt{n}$ (first inequality is saying $\|T\| \le \sqrt{n}$, second inequality is saying $\left\|T^{-1}\right\| \le 1$, $T\colon \ell_2^n \to X$.) John's Lemma is equivalent to: for every symmetric convex body $K \subset \mathbb{R}^n$, there exists an ellipsoid, $n^{-1/2}E \subset K \subset E$.

By compactness, there exists an ellipsoid E of minimal volume such that $K \subset E$. We will show $n^{-1/2}E \subset K$. By applying a linear bijection, WLOG $E = B_{\ell_2^n}$ [by replacing K with $T^{-1}(K)$]. Assume for contradiction that $n^{-1/2}E \not\subset K$. Then there exists $z \in \partial K = S_X$ such that $\|z\|_2 < \frac{1}{\sqrt{n}}$. By Hahn-Banach, there exists a linear functional $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(z) = 1$ and $|f(x)| \le 1$ for all $x \in K$. Let $H = \{x : f(x) + 1\}$. Then $z \in H$ and K is between H and $-H$. After applying a rotation, WLOG $H = \{x \in \mathbb{R}^n : x_1 = \frac{1}{c}\}$ for some $c > \sqrt{n}$ (as H contains a point with $\|\cdot\| < \frac{1}{\sqrt{n}}$). We still have $K \subset E = B_{\ell_2^n}$ and $K \subset \{x : |x_1| \le \frac{1}{c}\}$. Let $a > b > 0$, $E_{a,b} = \{x : a^2x_1^2 + \sum_{i=2}^n b^2x_i^2 \le 1\}$ which is the image of $B_{\ell_2^n}$ under the map with matrix diagonal $(\frac{1}{a}, \frac{1}{b}, \dots, \frac{1}{b})$. We have $\operatorname{vol}(E_{a,b}) = \frac{1}{ab^{n-1}} \operatorname{vol}(E)$. For $x \in K$, $a^2x_1^2 + \sum_{i=2}^n b^2x_i^2 = (a^2 - b^2)x_1^2 + \sum_{i=1}^n b^2x_i^2 \le \frac{a^2-b^2}{c^2} + b^2$ (using $K \subset E$). Need a, b such that $\frac{a^2-b^2}{c^2} + b^2 \le 1$ and

$ab^{n-1} > 1$. Then we would be done because $\text{vol}(E_{a,b}) < \text{vol}(E)$ and $K \subset E_{a,b}$ which contradicts the minimality of $\text{vol}(E)$.

For a given $0 < a < c$, set $b = \sqrt{\frac{c^2-a^2}{c^2-1}}$. Then $\frac{a^2-b^2}{c^2} + b^2 = 1$. Let $f(a) = ab^{n-1} = a\left(\frac{c^2-a^2}{c^2-1}\right)^{\frac{n-1}{2}}$. Then $f(1) = 1$,

$$\begin{aligned} f'(a) &= \left(\frac{c^2-a^2}{c^2-1}\right)^{\frac{n-1}{2}} + a\frac{n-1}{2}\frac{-2a}{c^2-1}\left(\frac{c^2-a^2}{c^2-1}\right)^{\frac{n-1}{2}} \\ &= \left(\frac{c^2-a^2}{c^2-1}\right)^{\frac{n-1}{2}-1}\left(\frac{c^2-a^2}{c^2-1} - \frac{(n-1)a^2}{c^2-1}\right) \\ &= \left(\frac{c^2-a^2}{c^2-1}\right)^{\frac{n-1}{2}-1}\left(\frac{c^2-na^2}{c^2-1}\right). \end{aligned}$$

Since $c^2 > n$, $f'(1) > 0$, there exists $a > 1$ such that $f(a) > f(1) = 1$. □

Definition. Let X, Y be metric spaces. A *Poincaré inequality* for functions $f\colon X \to Y$ is one of the form

$$\sum_{u,v\in X} a_{u,v}\Psi(d(f(u), f(v))) \geq \sum_{u,v\in X} b_{u,v}\Psi(d(f(u), f(v))), \qquad (*)$$

where a, b are $X \times X$ matrices, i.e. functions $a, b\colon X \times X \to \mathbb{R}^+$ of finite support, and Ψ is an increasing function $\mathbb{R}^+ \to \mathbb{R}^+$.

Define the *Poincaré ratio*

$$P_{a,b,\Psi}(X) = \frac{\sum_{u,v} b_{u,v}\Psi(d(u,v))}{\sum_{u,v} a_{u,v}\Psi(d(u,v))}, \qquad \text{whenever this is defined.}$$

Proposition. Let $1 \leq p < \infty$, $\Psi(t) = t^p$. Assume X, Y are metric spaces satisfying for some a, b the Poincaré inequality $(*)$ above for all functions $f\colon X \to Y$. Then $c_Y(X) \geq P_{a,b,\Psi}(X)^{1/p}$.

Proof. Let $f\colon X \to Y$ be a bilipschitz embedding [if there isn't any, then $c_Y(X) = \infty$]. Then

$$1 \geq \frac{\sum_{u,v} b_{u,v} d(f(u), f(v))^p}{\sum_{u,v} a_{u,v} d(f(u), f(v))^p} \geq \frac{1}{\text{dist}(f)^p}\frac{\sum_{u,v} b_{u,v} d(u,v)^p}{\sum_{u,v} a_{u,v} d(u,v)^p}$$

where the first inequality is by $(*)$. Hence $\text{dist}(f)^p \geq P_{a,b,\Psi}(X)^p$. Taking inf over all f gives the result. □

Example. In ℓ_2,

$$\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2,$$

for all $x_1, x_2, x_3, x_4 \in \ell_2$. This is a Poincaré inequality for functions $C_4 \to \ell_2$. Hence by the proposition above, $c_2(C_4) \geq \sqrt{\frac{2^2+2^2}{4}} = \sqrt{2}$. This can be achieved by the obvious embedding. So $c_2(C_4) = \sqrt{2}$.

To show that there is always a Poincaré inequality that gets arbitrarily close to the distortion, we need Hahn-Banach separation theorems (see Section 4).

Hahn-Banach Separation Theorems

To study Poincaré inequalities, we need to use the Hahn-Banach Separation Theorems. This section is a digression from Metric Embeddings.

Let X be a real vector space. A functional $p\colon X \to \mathbb{R}$ is *positive homogeneous* if $p(tx) = tp(x), \forall t \geq 0, \forall x \in X$, and *subadditive* if $p(x+y) \leq p(x)+p(y), \forall x, y \in X$. For example, a seminorm or a norm on X.

Theorem 4.1. Let X, p be as above. Let Y be a subspace of X, $g\colon Y \to \mathbb{R}$ a linear map such that $g(y) \leq p(y), \forall y \in Y$. Then there exists a linear map $f\colon X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \leq p(x)$ for all $x \in X$.

Proof. (This is similar to proof of Lemma 2.4). Let $P = \{(Z, h) : Z \leq X, h\colon Z \to \mathbb{R} \text{ linear}, Y \subset Z, h|_Y = g, h(z) \leq p(z), \forall z \in Z\}$. This is a poset with $(Z_1, h_1) \leq (Z_2, h_2) \iff Z_1 \subset Z_2$ and $h_2|_{Z_1} = h_1$. Note that $(Y, g) \in P$ so $P \neq \emptyset$. Given a chain $C = \{(Z_i, h_i) : i \in I\}$ in P (so C is linearly ordered) with $C \neq \emptyset$, then $Z = \bigcup_{i \in I} Z_i$ and $h\colon Z \to \mathbb{R}$ is defined by $h_{Z_i} = h_i, i \in I$ gives an upper bound (Z, h) for C. By Zorn's Lemma, P has a maximal element (W, k). We show that $W = X$, then we're done by taking $f = k$. Assume not. Fix $x_0 \in X \setminus W$ and let $W_1 = W + \mathbb{R}x_0$. Fix $\alpha \in \mathbb{R}$ and define $k_1\colon W_1 \to \mathbb{R}$ by $k_1(w + \lambda x_0) = k(w) + \lambda\alpha$ for $w \in W, \lambda \in \mathbb{R}$. We need α so that $k_1(w + \lambda x_0) \leq p(w + \lambda x_0$ for all $w \in W$ and $\lambda \in \mathbb{R}$. Then $(W, k) \lneqq (W_1, k_1)$, contradicting maximality of (W, k).

Since k_1 is linear and p is homogeneous, enough to get

$$k_1(w + x_0) \leq p(w + x_0), \qquad k_1(w - x_0) \leq p(w - x_0) \qquad \forall w \in W.$$

So we need

$$k(w) + \alpha \leq p(w + x_0), \qquad k(w) - \alpha \leq p(w - x_0) \qquad \forall w \in W.$$

So we need

$$k(z) - p(z - x_0) \leq \alpha \leq p(w + x_0) - k(w) \qquad \forall w, z \in W.$$

We need $k(z) - p(z - x_0) \leq p(w + x_0) - k(w), \forall w, z \in W$. Then $\alpha = \inf_{w \in W}(p(w + x_0) - k(w))$ will do. But $k(z) + k(w) = k(z + w) \leq p(z + w) = p(z - x_0 + w + x_0) \leq p(z - x_0) + p(w + x_0), \forall w, z \in W$. ☐

Corollary 4.2. Let X be a real normed space.

(i) If Y is a subspace and $g \in Y^*$ then there exists $f \in X^*$ s.t. $f|_Y = g$ and $\|f\| = \|g\|$. [Hahn-Banach Extension Theorem]

(ii) Given $x_0 \in X$, $x_0 \neq 0$, there exists $f \in S_{X^*}$ such that $f(x) = \|x_0\|$. [Norming functional for x_0]

Proof. (i) Let $p(x) = \|g\| \|x\|$ for $x \in X$. Then p is a seminorm. We have $g(y) \leq p(y)$ for all $y \in Y$. By Theorem 2, there exists a linear $f\colon X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \leq \|g\| \|x\|$ for all $x \in X$. Apply this to $-x$ to get $-f(x) = f(-x) \leq \|g\| \|x\|$. So $|f(x)| \leq \|g\| \|x\|$ for all $x \in X$. So $f \in X^*$ and $\|f\| \leq \|g\|$. Since $f|_Y = g$, $\|f\| = \|g\|$.

(ii) Define $g\colon Y := \mathbb{R}x_0 \to \mathbb{R}$ by $g(\lambda x_0) = \lambda \|x_0\|$ for $\lambda \in \mathbb{R}$. Then $g \in Y^*$ and $\|g\| = 1$. So by (i), there exists $f \in S_{X^*}$ such that $f|_Y = g$, and so $f(x_0) = \|x_0\|$.

☐

Remark 4.3. If Z is a complex vector space, let $Z_{\mathbb{R}}$ be Z viewed as a real vector space. Then for a complex normed space X, the map $(X^*)_{\mathbb{R}} \to (X_{\mathbb{R}})^*$, $f \mapsto \operatorname{Re} f$ is an isometric isomorphism. Thus (i) follows in the complex case.

Given a normed space X and a convex subset C of X with $0 \in \operatorname{Int} C$, the *Minkowski functional* of C is $\mu_C \colon X \to \mathbb{R}$ defined by

$$\mu_C(x) = \inf\{t > 0 : x \in tC\}.$$

This is well-defined: given $x \in X$, $\frac{x}{n} \to 0 \in \operatorname{Int} C$, so $\exists n, \frac{x}{n} \in C$, i.e. $x \in nC$. Example: If $C = B_X$, then $\mu_C = \|\cdot\|$ as $x \in tB_X \iff \|x\| \le t$.

Lemma 4.4. Let X, C be as above. Then μ_C is positive homogeneous and subadditive. Moreover,

$$\{x \in X : \mu_C(x) < 1\} \subset C \subset \{x \in X : \mu_C(x) \le 1\},$$

with equality in the first inclusion if C is open, and with equality in the second inclusion if C is closed.

Proof. For positive homogeneity, e need $\mu_C(tx) = t\mu_C(x)$ for all $t \ge 0$ and $x \in X$. For $t = 0$, we need $\mu_C(0) = 0$. This is true since $x \in tC$ for all $t > 0$. If $t > 0$, then for any $s > 0$, $tx \in sC \iff x \in \frac{s}{t}C$, so $\mu_C(tx) = t\mu_C(x)$.

For subadditivity, fix $x, y \in X$ and let $s > \mu_C(x), t > \mu_C(y)$. Then by definition, there exists s', $\mu_C(x) \le s' < s$ such that $x \in s'C$. Then $\frac{x}{s} = \frac{s'}{s}\frac{x}{s'} + (1 - \frac{s'}{s})0 \in C$, since C is convex. So $x \in sC$. Also $y \in tC$. Thus $\frac{x+y}{s+t} = \frac{s}{s+t}\frac{x}{s} + \frac{t}{s+t}\frac{y}{t} \in C$. This shows $\mu_C(x+y) \le s + t$. Taking inf over all s, t we get subadditivity.

If $1 > \mu_C(x)$, then by above $x \in C$, showing the first inclusion. If $x \in C$, then $\mu_C(x) \le 1$ by definition, showing the second inclusion. Assume C is open. If $x \in C$, then since $(1 + \frac{1}{n})x \to x$ and C is open, then there exists n with $(1 + \frac{1}{n})x \in C$, i.e. $x \in \frac{n}{n+1}C$, so $\mu_C(x) \le \frac{n}{n+1} < 1$. Now assume C is closed and $\mu_C(x) \le 1$. Then $\mu_C(\frac{n}{n+1}x) \le \frac{n}{n+1} < 1$ so $\frac{n}{n+1}x \in C$ for all $n \in \mathbb{N}$. Since $\frac{n}{n+1}x \to x$ and C is closed, $x \in C$. □

Theorem 4.5. Let X be a real normed space, and let C be an open convex subset of X with $0 \in C$. For $x_0 \in X \setminus C$, there exists $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$. (Note that $f \neq 0$.)

Proof. Define $Y = \mathbb{R}x_0$ and $g \colon Y \to \mathbb{R}$ by $g(\lambda x_0) = \lambda\mu_C(x_0)$. Then g is linear and for $\lambda \ge 0$, $g(\lambda x_0) = \lambda\mu_C(x_0) = \mu_C(\lambda x_0)$, and for $\lambda < 0$, $g(\lambda x_0) = \lambda\mu_C(x_0) \le 0 \le \mu_C(\lambda x_0)$. By Lemma 4 and Theorem 2, there exists a linear map $f \colon X \to \mathbb{R}$ such that $f|_Y = g$ and $f(x) \le \mu_C(x)$ for all $x \in X$. Since $x_0 \notin C$, $\mu_C(x_0) \ge 1$. So for all $x \in C$, $f(x) \le \mu_C(x) < 1 \le \mu_C(x_0) = f(x_0)$ [here we used C is open]. Since $0 \in C$, C open, $\exists \delta > 0$ such that $\delta B_X \subset C$. So $f(x) \le 1$ on δB_X, but this is symmetric, so $|f(x)| \le 1$. So $f \in X^*$. □

Remark. If Lemma 4, if C is symmetric, then μ_C is a seminorm. If in addition, C is bounded, then μ_C is a norm [we used this in the proof of Theorem 1].

Corollary 4.6 (The Hahn-Banach Separation Theorems)**.** Let A, B be non-empty, disjoint convex sets in a normed space X.

(i) If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \leq f(y)$ for all $x \in A$, for all $y \in B$.

(ii) If A is compact, and B is closed, then $\exists f \in X^*$ and $\alpha \in \mathbb{R}$ such that $\sup_A f < \alpha < \inf_B f$.

Remark. In both cases, the hyperplane $\{x \in X : f(x) = \alpha\}$ separates A and B.

Proof. (i) Fix $a_0 \in A, b_0 \in B$. Let $C = A - B - a_0 + b_0$, $x_0 = -(a_0 - b_0)$. Then C is convex and open, $0 \in C$ and $x_0 \notin C$ since $A \cap B = \emptyset$. By Theorem 5, $\exists f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$. So $f(x - y + x_0) < f(x_0)$ for all $x \in A, y \in B$, i.e., $f(x) < f(y)$ for all $x \in A, y \in B$. Let $\alpha = \inf_B f$. So certainly we have $f(y) \geq \alpha$ for all $y \in B$. Also, $f(x) \leq \alpha$ for all $x \in A$. Since $f \neq 0$, we can fix $u \in X$ such that $f(u) > 0$. For $x \in A$, since A is open, $\exists n \in \mathbb{N}$ such that $x + \frac{1}{n}u \in A$. Then $f(x) < f(x + \frac{1}{n}u) \leq \alpha$.

(ii) For $x \in A$, $d(x, B) > 0$ since B is closed and $x \notin B$. Since A is compact, $\delta = \inf_{x \in A} d(x, B) > 0$. Then $A' = \{x \in X : d(x, A) < \delta\}$ is an open, convex set with $A' \cap B = \emptyset$. [If $d(x, A), d(y, A) < \delta$ then $\exists u, v \in A$, $\|x - u\|, \|y - v\| < \delta$ and then $\forall t \in (0, 1)$,

$$\|((1-t)x + ty) - ((1-t)u + tv)\| < \delta,$$

$(((1-t)u + tv) \in A)$, so $(1-t)x + ty \in A'$]. By (i), $\exists f \in X^*, \exists \beta \in \mathbb{R}$ such that $f(x) < \beta \leq f(y)$ for all $x \in A', y \in B$. As A is compact, $\sup_A f < \beta \leq \inf_B f$.

□

Poincaré Inequalities

Now we can show that Poincaré inequalities are worth studying because they get arbitrarily close to the distortion of f.

Theorem 4.7. Let $1 \leq p < \infty$ and X be a finite metric space. Then

$$c_p(X) = \sup \left(P_{a,b,t^p}(X)\right)^{1/p},$$

where the sup is over all non-negative, non-trivial $X \times X$ matrices a, b for which the Poincaré inequality

$$\sum_{u,v \in X} a_{u,v} \|f(u) - f(v)\|_p^p \geq \sum_{u,v \in X} b_{u,v} \|f(u) - f(v)\|_p^p \qquad (*)$$

holds.

Proof. From Proposition 2, $c_p(X) \geq \sup \left(P_{a,b,t^p}(X)\right)^{1/p}$. Taking $a_{u,v} = b_{u,v} = 1$ for all u, v, $(*)$ holds, and $P_{a,b,t^p}(X) = 1$, so if $c_p(X) = 1$ then we are done.

Now assume $1 < C < c_p(X)$. Let $X = \{x_1, ..., x_n\}$. Let

$$B = \left\{ \left(\|f(x_i) - f(x_j)\|_p^p \right)_{1 \leq i < j \leq n} : f \colon X \to L_p \right\} \subset \mathbb{R}^N,$$

where $N = \binom{n}{2}$. From proof of Theorem 2.7, we know B is a cone, and in particular, B is convex. Also $B \neq \emptyset$ because $0 \in B$. Let

$$A = \left\{ (\theta_{ij})_{1 \le i < j \le n} \in \mathbb{R}^N : \exists r > 0, r d(x_i, x_j)^p < \theta_{ij} < r C^p d(x_i, x_j)^p, \forall i, j \right\}.$$

Then A is open, convex and non-empty since $C > 1$. Since $C < c_p(X)$, we have $A \cap B = \emptyset$. By Corollary 6, there exists a linear map $c\colon \mathbb{R}^N \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $c(\theta) < \alpha \le c(\varphi)$ for all $\theta \in A, \varphi \in B$. We have $c = (c_{ij})_{1 \le i < j \le n}$ where $c(\theta) = \sum_{1 \le i < j \le n} c_{ij} \theta_{ij}$. Since $0 \in B$, $\alpha \le 0$. By continuity, $c(\theta) \le \alpha$ for all $\theta \in \bar{A}$, and $0 \in \bar{A}$, so $0 \le \alpha$. Hence $\alpha = 0$. So $c(\theta) \le 0 \le c(\varphi)$ for all $\theta \in \bar{A}, \varphi \in B$. Let $a_{ij} = \max(c_{ij}, 0)$, $b_{ij} = \max(-c_{ij}, 0)$. So $c_{ij} = a_{ij} - b_{ij}$. We have

$$\sum c_{ij} \left\| f(x_i) - f(x_j) \right\|_p^p \ge 0,$$

for all $f\colon X \to L_p$, i.e.

$$\sum_{1 \le i < j \le n} a_{ij} \left\| f(x_i) - f(x_j) \right\|_p^p \ge \sum_{1 \le i < j \le n} b_{ij} \left\| f(x_i) - f(x_j) \right\|_p^p,$$

for all $f\colon X \to L_p$.

Let

$$\theta_{ij} = \begin{cases} C^p d(x_i, x_j)^p & \text{if } c_{ij} \ge 0, \\ d(x_i, x_j)^p & \text{if } c_{ij} < 0. \end{cases}$$

Then $\theta = (\theta_{ij}) \in \bar{A}$, so

$$0 \ge c(\theta) = \sum_{ij} a_{ij} C^p d(x_i, x_j)^p - \sum_{ij} b_{ij} d(x_i, x_j)^p.$$

Thus $P_{a,b,t^p}(X) \ge C^p$. □

Hamming Cube

Recall $H_n = \{0,1\}^n$, which is a graph: $x = (x_i), y = (y_i)$ are joined by an edge $\iff$ $x_i \neq y_i$ for exactly one value of i. So H_n is a metric space with the graph distance d:

$$d(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

So H_n is isometrically a subset of ℓ_1^n.

H_n is also a probability space with the uniform distribution μ: $\mu(\{x\}) = 2^{-n}$.

We think of $\{0,1\}$ as the field $\mathbb{F}_2$. Then H_n is the n-dimensional vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$. So in particular, H_n is an abelian group.

Notation. Let $(e_i)_{i=1}^n$ is the standard basis of $H_n = \mathbb{F}_2^n$. For $j = 1, ..., n$, let $r_j\colon H_n \to \mathbb{R}$, $r_j(x) = (-1)^{x_j}$. This is the jth *Rademacher function*. Note that $r_1, ..., r_n$ are iid random variables on (H_n, μ) with $\{\pm 1\}$-valued $\texttt{Rademacher}(\frac{1}{2})$ distribution. For $A \subset \{1, ..., n\}$, we define $w_A\colon H_n \to \mathbb{R}$, $w_A = \prod_{j \in A} r_j$. These are the *Walsh functions*. These are the characters of H_n, i.e. abelian group homomorphisms $H_n \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 4.8. The Walsh functions form an orthonormal basis of $L_2(H_n, \mu)$.

Proof. We have $r_j^2 = 1$, so for $A, B \subset \{1, ..., n\}$, $w_A w_B = \prod_{j \in A} r_j \prod_{k \in B} r_k = \prod_{j \in A \triangle B} r_j = w_{A \triangle B}$. So if $A = B$, $\langle w_A, w_A \rangle = \int_{H_n} w_\emptyset \, d\mu = 1$. If $A \neq B$, by independence, $\langle w_A, w_B \rangle = \int_{H_n} w_{A \triangle B} \, d\mu = \prod_{j \in A \triangle B} \int_{H_n} r_j \, d\mu = 0$. Alternatively, shifting is a measure-preserving transformation. Fix $j \in A \triangle B$, $\int_{H_n} w_{A \triangle B}(x) \, d\mu(x) = \int_{H_n} w_{A \triangle B}(x + e_j) \, d\mu(x) = -\int_{H_n} w_{A \triangle B}(x) \, d\mu(x)$. We're done as $\dim L_2(H_n, \mu) = 2^n$. □

Definition. For $f \colon H_n \to \mathbb{R}$, we let $\hat{f}_A = \langle f, w_A \rangle = \int_{H_n} f w_A \, d\mu$ for $A \subset \{1, ..., n\}$. These are the *Fourier coefficients* of f with respect to this orthonormal basis. More generally, for a Banach space X and $f \colon H_n \to X$, we define $\hat{f}_A = \int_{H_n} f(x) w_A(x) \, d\mu(x)$, $A \subset \{1, ..., n\}$. Normally this would involve the Bochner integral, but here everything is finite, so this is just a summation.

Lemma 4.9. (a) For any $f \in L_2(H_n, \mu)$ we have

$$f(x) = \sum_{A \subset \{1, ..., n\}} \hat{f}_A w_A(x), \qquad x \in H_n,$$

$$\int_{H_n} |f(x)|^2 \, d\mu(x) = \sum_{A \subset \{1, ..., n\}} |\hat{f}_A|^2, \qquad \text{Parseval's identity}$$

(b) If X is a Banach space, then for all $f \colon H_n \to X$ we have

$$f(x) = \sum_{A \subset [n]} \hat{f}_A w_A(x), \qquad x \in H_n.$$

If in addition X is a Hilbert space, then

$$\int_{H_n} \|f(x)\|^2 \, d\mu(x) = \sum_{A \subset [n]} \left\| \hat{f}_A \right\|^2 \qquad \text{Parseval's identity.}$$

Proof. (a) Follows from Lemma 8. (b) Fix $\varphi \in X^*$. Then

$$\varphi(\hat{f}_A) = \int_{H_n} \varphi(f(x)) w_A(x) \, d\mu(x) = \widehat{\varphi \circ f}_A \qquad \forall A \subset [n].$$

So for any $x \in H_n$, we have, by (a),

$$\varphi(f(x)) = \sum_A \widehat{\varphi \circ f}_A w_A(x) = \varphi(\sum_A \hat{f}_A w_A(x)).$$

This holds for all $\varphi \in X^*$, so by Hahn-Banach, $f(x) = \sum_A \hat{f}_A w_A(x)$. True for all $x \in H_n$.

If X is a Hilbert space, then WLOG $\dim X < \infty$. Fix an orthonormal basis $v_1, ..., v_k$ of X. Then for $1 \leq j \leq k$, let $f_j(x) = \langle f(x), v_j \rangle$. By above, taking $\varphi(u) = \langle u, v_j \rangle$, $\hat{f_j}_A = \langle \hat{f}_A, v_j \rangle$ Then by Parseval in X, in $L_2(H_n, \mu)$, and in X respectively,

$$\begin{aligned} \int_{H_n} \|f(x)\|^2 \, d\mu(x) &= \int_{H_n} \sum_{j=1}^k |f_j(x)|^2 \, d\mu(x) = \sum_{j=1}^k \sum_A |\hat{f_j}_A|^2 \\ &= \sum_A \sum_j |\langle \hat{f}_A, v_j \rangle|^2 = \sum_A \|\hat{f}_A\|^2. \end{aligned}$$

□

Definition. For each $1 \leq j \leq n$, we define a *difference operator* ∂_j as follows. For a Banach space X and $f\colon H_n \to X$, let $\partial_j f\colon H_n \to X$ be defined as

$$(\partial_j f)(x) = \frac{f(x+e_j) - f(x)}{2}.$$

Lemma 4.10. (i) For $1 \leq j \leq n$, $A \subset [n]$,

$$\partial_j w_A(x) = \begin{cases} -w_A(x) & j \in A \\ 0 & j \notin A. \end{cases}$$

(ii) For a Banach space X and $f\colon H_n \to X$,

$$\widehat{\partial_j f}_A = \begin{cases} -\hat{f}_A & j \in A \\ 0 & j \notin A. \end{cases}$$

(iii) If X is a Hilbert space, then for $f\colon H_n \to X$,

$$\sum_{j=1}^n \int_{H_n} \|\partial_j f(x)\|^2 \, d\mu(x) = \sum_A |A| \|\hat{f}_n\|^2.$$

Proof. (i) We have

$$r_i(x+e_j) = \begin{cases} -r_i(x) & j = i \\ r_i(x) & j \neq i. \end{cases}$$

So

$$w_A(x+e_j) = \prod_{i \in A} r_i(x+e_j) = \begin{cases} -w_A(x) & j \in A \\ w_A(x) & j \notin A. \end{cases}$$

Hence result follows.

(ii) This is integration by parts:

$$\begin{aligned}
(\widehat{\partial_j f})_A &= \int_{H_n} (\partial_j f)(x) w_A(x) \, d\mu(x) \\
&= \frac{1}{2} \int_{H_n} f(x+e_j) w_A(x) \, d\mu(x) - \frac{1}{2} \int_{H_n} f(x) w_A(x) \, d\mu(x) \\
&= \frac{1}{2} \int_{H_n} f(x) w_A(x+e_j) \, d\mu(x) - \frac{1}{2} \int_{H_n} f(x) w_A(x) \, d\mu(x) \\
&= \int_{H_n} f(x) (\partial_j w_A)(x) \, d\mu(x) \\
&= \begin{cases} -\hat{f}_A & j \in A \\ 0 & j \notin A. \end{cases}
\end{aligned}$$

(iii) We use Parseval:

$$\begin{aligned}
\sum_{j=1}^n \int_{H_n} \|\partial_j f(x)\|^2 \, d\mu(x) &= \sum_{j=1}^n \sum_A \|(\widehat{\partial_j f})_A\|^2 \\
&= \sum_A \sum_j \|(\widehat{\partial_j f})_A\|^2 \\
&= \sum_A |A| \|\hat{f}_A\|^2,
\end{aligned}$$

as required. □

Theorem 4.11 (Poincaré inequality for L_2-valued functions on H_n)**.** Let $e = e_1 + e_2 + ... + e_n = (1, 1, ..., 1)$. Then for all $f\colon H_n \to L_2$, we have

$$\int_{H_n} \|f(x+e) - f(x)\|^2 \, d\mu(x) \leq 4 \sum_{j=1}^{n} \int_{H_n} \|(\partial_j f)(x)\|^2 \, d\mu(x).$$

Proof. For $A \subset [n]$, $w_A(x+e) = (-1)^{|A|} w_A(x)$. So

$$\begin{aligned}
&\int_{H_n} \|f(x+e) - f(x)\|^2 \, d\mu(x) \\
&\qquad = \int_{H_n} \left\| \sum_A \hat{f}_A w_A(x+e) - \sum_A \hat{f}_A w_A(x) \right\|^2 d\mu(x) \quad \text{(by Lemma 9)} \\
&\qquad = 4 \int_{H_n} \left\| \sum_{A:|A| \text{ odd}} \hat{f}_A w_A(x) \right\|^2 d\mu(x) \\
&\qquad = 4 \sum_{A:|A| \text{ odd}} \|\hat{f}_A\|^2 \quad \text{(by Lemma 9)} \\
&\qquad \leq 4 \sum_{|A|} |A| \|\hat{f}_A\|^2 \\
&\qquad = 4 \sum_{j=1}^{n} \int_{H_n} \|(\partial_j f)(x)\|^2 \, d\mu(x).
\end{aligned}$$

□

Corollary 4.12. $c_2(H_n) = \sqrt{n}$.

Remark. $|H_n| = 2^n$, so $c_2(H_n) = \sqrt{\log |H_n|}$. Compare with the upper bound $c_2(H_n) \lesssim \log |H_n|$ in Bourgain's embedding theorem.

Proof of Corollary 12. $H_n \subset \ell_2^n$ in the obvious way which gives $c_2(H_n) \leq \sqrt{n}$. By Proposition 2, a lower bound on $c_2(H_n)$ is obtained from the Poincaré ratio

$$\frac{\int_{H_n} d(x+e, x)^2 \, d\mu(x)}{4 \sum_{j=1}^{n} \int_{H_n} \frac{d(x+e_j, x)^2}{4} \, d\mu(x)} = \frac{n^2}{n} = n,$$

so $c_2(H_n) \geq \sqrt{n}$. □

From now on, think of H_n as the n-dimensional vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$.

Theorem 4.13. For every $f\colon \mathbb{F}_2^n \to L_2$ we have the Poincaré inequality:

$$\int_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \|f(x) - f(y)\|_{L_2}^2 \, d\mu(x) \, d\mu(y) \leq \frac{2}{\max\{|A| : A \neq \emptyset, \hat{f}_A \neq 0\}} \sum_{j=1}^{n} \int_{\mathbb{F}_2^n} \|\partial_j f(x)\|^2 \, d\mu(x).$$

Proof. Without loss of generality, after replacing f with $f - \hat{f}_\emptyset w_\emptyset$, can assume $\hat{f}_\emptyset = 0$ (recall $w_\emptyset = 1$). Then by Parseval,

$$\begin{aligned} LHS &= \int_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \|f(x)\|^2 + \|f(y)\|^2 - 2\langle f(x), f(y)\rangle \, d\mu(x)\, d\mu(y) \\ &= 2\sum_A \|\hat{f}_A\|^2 - 2\int_{\mathbb{F}_2^n} \left\langle \int_{\mathbb{F}_2^n} f(x)\, d\mu(x), f(y) \right\rangle d\mu(y) \\ &= 2\sum_A \|\hat{f}_A\|^2 - 2\int_{\mathbb{F}_2^n} \langle \hat{f}_\emptyset, f(y)\rangle \, d\mu(y) \\ &= 2\sum_A \|\hat{f}_A\|^2. \end{aligned}$$

By Lemma 10,

$$\sum_{j=1}^n \int_{\mathbb{F}_2^n} \|\partial_j f(x)\|^2 \, d\mu(x) = \sum_A |A| \|\hat{f}_A\|^2 \geq \min\{|A| : A \neq \emptyset, \hat{f}_A \neq 0\} \sum_A \|\hat{f}_A\|^2.$$

□

Definition. A *linear code* of $\mathbb{F}_2^n$ is a subspace C of $\mathbb{F}_2^n$. We let $d(C) = \min\{d(x, 0) : x \in C, x \neq 0\} = d(0, C \setminus \{0\})$. For $x = (x_i), y = (y_i)$ in $\mathbb{F}_2^n$, let $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ (operations in $\mathbb{F}_2^n$). This is a symmetric bilinear form, but $\langle x, x\rangle = 0$ does not imply $x = 0$. For a subset $S \subset \mathbb{F}_2^n$, let $S^\perp = \{x \in \mathbb{F}_2^n : \langle x, s\rangle = 0, \forall s \in S\}$.

Linear Codes

Lemma 4.14. For a linear code C, $\dim C + \dim C^\perp = n$ and $C^{\perp\perp} = C$.

Proof. Let $m = \dim C$ and $v_1, ..., v_m$ be a basis of C. Define $\theta \colon \mathbb{F}_2^n \to \mathbb{F}_2^m$ by $\theta(x) = (\langle x, v_i\rangle)_{i=1}^m$. Then $\ker\theta = C^\perp$ and so $n = \dim C^\perp + \dim \operatorname{im}\theta$. We need θ to be onto. For $1 \leq j \leq m$, let $f \colon \mathbb{F}_2^n \to \mathbb{F}_2$ be a linear map such that $f(v_i) = \delta_{ij}$ (Kronecker delta). Let $y_j = f(e_j)$ for $1 \leq j \leq n$ and $y = (y_j)$. Then $f(x) = \sum_{j=1}^n x_j f(e_j) = \langle x, y\rangle$, so $\theta(y) = (f(v_j))_{j=1}^n = i$th standard basis vector of $\mathbb{F}_2^m$. So $n = \dim C^\perp + m = \dim C^\perp + \dim C$. For the final part: from definition, $C \subset C^{\perp\perp}$, and $\dim C^{\perp\perp} = n - \dim C^\perp = \dim C$, so $C = C^{\perp\perp}$. □

Lemma 4.15. There exists $\delta \in (0, \frac{1}{2}), \exists N \in \mathbb{N}, \forall n \geq N, (m+1)\binom{n}{m} \leq 2^{n/8}$ where $m = \lfloor \delta n \rfloor$.

Proof. First choose $\delta \in (0, \frac{1}{2})$ such that $\delta(2 + \log(\frac{2}{\delta})) < \frac{\log 2}{8}$. Then choose $N \in \mathbb{N}$ such that $\lfloor \delta n \rfloor \geq \frac{\delta n}{2}, \forall n \geq N$. Let $n \geq N$ and $m = \lfloor \delta n \rfloor$. If $m = 0$ then we are done, so assume $m \geq 1$. Then $\binom{n}{m} = \frac{n(n-1)(n-2)\dots(n-m+1)}{m!} \leq \frac{n^m}{m!}$. For the denominator use $\log(m!) = \sum_{j=1}^m \log(j) \geq \int_1^m \log x \, dx = [x\log x - x]_1^m = m\log m - m + 1 \geq m\log m - m$. So $\binom{n}{m} \leq (\frac{en}{m})^m$ and $(m+1)\binom{n}{m} \leq (m+1)(\frac{en}{m})^m$.

Now

$$\begin{aligned}\log\left((m+1)\binom{n}{m}\right) &\le \log(m+1) + m(1+\log(n/m)) \\ &\le m(2+\log(n/m)) && (\log x \le x-1, \forall x>0) \\ &\le \delta n(2+\log(2/\delta)) && (\frac{\delta n}{2} \le m = \lfloor \delta n \rfloor \le \delta n) \\ &\le \frac{\log 2}{8} n.\end{aligned}$$

Thus $(m+1)\binom{n}{m} \le 2^{n/8}$. □

Lemma 4.16. *$\exists \alpha > 0, \forall n \in \mathbb{N}, \exists$ linear code C in $\mathbb{F}_2^n$ with $\dim C \ge \frac{n}{4}$ and $d(C) \ge \alpha n$.*

Proof. Let δ, N be as in Lemma 15. If $1 \le n \le N$, choose any C with $\dim C \ge \frac{n}{4}$. Then $d(C) \ge 1 \ge \frac{1}{N} n$. Now let $n > N$. We show there exists a linear code C in $\mathbb{F}_2^n$ such that $\dim C \ge \frac{n}{4}$ and $d(C) \ge \delta n$. So $\alpha = \min(\frac{1}{N}, \delta)$ will do.

We choose C greedily. Assume that for some k, $1 \le k < \frac{n}{4}$ we have a linear code C_k with $\dim C_k = k$ and $d(C_k) \ge \delta n$. For $k = 1$ this holds. We seek a suitable $x \in \mathbb{F}_2^n \setminus C_k$ such that putting $C_{k+1} = \operatorname{span}(C_k \cup \{x\}) = C_k \cup (C_k + x)$, we have $d(C_{k+1}) \ge \delta n$. Once we find such x, we continue inductively. Taking $C = C_{\lceil n/4 \rceil}$ will complete the proof.

We estimate from above the number of unsuitable vectors x. For $v \in C_k$,

$$\begin{aligned}|\{x : d(v+x, 0) < \delta n\}| &= |\{x : d(x,0) < \delta n\}| \\ &= \sum_{0 \le \ell < \delta n} \binom{n}{\ell} \\ &\le (m+1)\binom{n}{m},\end{aligned}$$

where $m = \lfloor \delta n \rfloor$. Note in the range $0 \le \ell \le \frac{n}{2}$, $\binom{n}{\ell}$ is increasing, and $\delta < \frac{1}{2}$. It follows that

$$\begin{aligned}|\{x \in \mathbb{F}_2^n : \exists v \in C_k, d(x+v,0) < \delta n\}| &= \left| \bigcup_{v \in C_k} \{x \in \mathbb{F}_2^n : d(x+v,0) < \delta n\} \right| \\ &\le 2^k (m+1)\binom{n}{m}.\end{aligned}$$

If $2^k(m+1)\binom{n}{m} < 2^n - 2^k$ then there is a suitable x, i.e. we need $(m+1)\binom{n}{m} < 2^{n-k} - 1$. Now $2^{n-k} - 1 > 2^{3n/4} - 1 \ge 2^{n/8}$, so we are done by choice of δ, N. □

From now on, C will be an arbitrary linear code in $\mathbb{F}_2^n$. Let $q \colon \mathbb{F}_2^n \to \mathbb{F}_2^n / C^\perp$ be the quotient map. Let $\tilde{\mu}$ be the image measure induced by μ and q: $\tilde{\mu}(E) = \mu(q^{-1}(E))$. Let ρ be the quotient metric on $\mathbb{F}_2^n / C^\perp$: $\rho(q(x), q(y)) = d(x + C^\perp, y + C^\perp) = d(x-y, C^\perp) = \min_{v \in C^\perp} d(x-y, v)$.

Lemma 4.17. *For every $h \colon \mathbb{F}_2^n / C^\perp \to L^1$ and for every $A \subset [n]$ with $A \ne \emptyset$ and $|A| < d(C)$ we have $\hat{f}_A = 0$ where $f = h \circ q$.*

Proof. Let $v = \sum_{i \in A} e_i$. Then $v \neq 0$ since $A \neq \emptyset$ and $d(v, 0) = |A| < d(C)$¿ So $v \notin C = C^{\perp\perp}$ (Lemma 14). So $\exists w \in C^\perp$ such that $\langle v, w\rangle \neq 0$, i.e. $\langle v, w\rangle = 1$. Now

$$\begin{aligned}
\hat{f}_A &= \int_{\mathbb{F}_2^n} f(x) w_A(x)\, d\mu(x) \\
&= \int_{\mathbb{F}_2^n} f(x+w) w_A(x+w)\, d\mu(x) \\
&\qquad\qquad \text{(translation invariance of } \mu) \\
&= \int_{\mathbb{F}_2^n} f(x) \prod_{j \in A} r_j(x+w)\, d\mu(x) \\
&\qquad\qquad (w \in C^\perp \text{ so } f(x+w) = hq(x+w) = hq(x) = f(x)) \\
&= \int_{\mathbb{F}_2^n} f(x) \prod_{j \in A} (-1)^{w_j} r_j(x)\, d\mu(x) \\
&= \int_{\mathbb{F}_2^n} f(x) (-1)^{\langle v, w\rangle} w_A(x)\, d\mu(x) \\
&= -\hat{f}_A.
\end{aligned}$$

Hence $\hat{f}_A = 0$. □

Theorem 4.18 (Poincaré inequality for L_1-valued functions on $\mathbb{F}_2^n/C^\perp$)**.** For every $h\colon \mathbb{F}_2^n/C^\perp \to L_1$ we have

$$\int_{(\mathbb{F}_2^n/C^\perp)^2} \|h(u) - h(v)\|_{L_1}\, d\tilde{\mu}(u)\, d\tilde{\mu}(v) \leq \frac{1}{d(C)} \sum_{j=1}^n \int_{\mathbb{F}_2^n/C^\perp} \|\partial_j h(u)\|_{L_1}\, d\tilde{\mu}(u) \qquad (*)$$

where

$$\partial_j h(u) = \frac{h(u + q(e_j)) - h(u)}{2},$$

and $u \in \mathbb{F}_2^n/C^\perp$.

Proof. Let $f = h \circ q$. Then $(*)$ is equivalent to

$$\int_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \|f(x) - f(y)\|_{L_1}\, d\mu(x)\, d\mu(y) \leq \frac{1}{d(C)} \sum_{j=1}^n \int_{\mathbb{F}_2^n} \|\partial_j f(x)\|_{L_1}\, d\mu(x).$$

From (proof of) Proposition 1.7, there exists a map $T\colon L_1 \to L_2$ such that

$\|Ta - Tb\|_{L_2} = \|a-b\|_{L_1}^{1/2}$. Now

$$\begin{aligned}
&\int\limits_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \|f(x) - f(y)\|_{L_1} \, d\mu(x)\, d\mu(y) \\
&\qquad = \int_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \|Tf(x) - Tf(y)\|_{L_2}^2 \, d\mu(x)\, d\mu(y) \\
&\qquad \le \frac{2}{\min\{|A| : A \neq \emptyset, \hat{f}_A \neq 0\}} \sum_{j=1}^n \int_{\mathbb{F}_2^n} \|\partial_j Tf(x)\|_{L_2}^2 \, d\mu(x) \quad \text{(Theorem 13)} \\
&\qquad \le \frac{2}{d(C)} \sum_{j=1}^n \int_{\mathbb{F}_2^n} \|\partial_j Tf(x)\|_{L_2}^2 \, d\mu(x) \quad \text{(Lemma 17)} \\
&\qquad = \frac{1}{d(C)} \sum_{j=1}^n \int_{\mathbb{F}_2^n} \|\partial_j f(x)\|_{L_1}^2 \, d\mu(x)
\end{aligned}$$

since $\|\partial_j Tf(x)\|_{L_2}^2 = \frac{\|Tf(x+e_j) - Tf(x)\|_{L_2}^2}{4} = \frac{\|f(x+e_j)-f(x)\|_{L_1}}{4} = \frac{1}{2}\|\partial_j f(x)\|_{L_1}$. □

Lemma 4.19. $\exists \beta > 0, \forall n \in \mathbb{N}$, if $\dim C \ge \frac{n}{4}$ then $\forall x \in \mathbb{F}_2^n$,

$$\mu(\{y : \rho(qx, qy) \ge \beta n\}) \ge \frac{1}{2}.$$

Proof. Let n, δ be as in Lemma 15. WLOG $N \ge 8$. WLOG $x = 0$. For $1 \le n \le N$, $\mu(\{y : \rho(qy, 0) \ge \frac{n}{N}\}) = \mu(\mathbb{F}_2^n \setminus C^\perp) = \frac{2^n - |C^\perp|}{2^n}$. From Lemma 14, $\dim C^\perp = n - \dim C \le n-1$, so $\frac{2^n - |C^\perp|}{2^n} \ge \frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2}$. Now let $n > N$. For $v \in C^\perp$, consider

$$|\{y : d(v,y) < \delta n\}| \le \sum_{0 \le \ell < \delta n} \binom{n}{\ell} \le (m+1)\binom{n}{m},$$

where $m = \lfloor \delta n \rfloor$. So

$$\begin{aligned}
|\{y : \exists v \in C^\perp, d(y,v) < \delta n\}| &= |\{y : \rho(qy, 0) < \delta n\}| \\
&\le 2^{\dim C^\perp} (m+1) \binom{n}{m} \\
&\le 2^{3n/4} 2^{n/8} \le \frac{1}{2} 2^n.
\end{aligned}$$

(Here we use $n > N \ge 8$). So $\mu(\{y : \rho(qy, 0) \ge \delta n\}) \ge \frac{1}{2}$. So $\beta = \min(\delta, \frac{1}{N})$ works. □

Theorem 4.20. $\exists \eta > 0, \exists$ sequence (X_n) of metric spaces such that $|X_n| \to \infty$ and $c_1(X_n) \ge \eta \log |X_n|$.

Remark. Recall $c_2(X) \ge c_1(X)$ for any finite metric space. So Theorem 20 says that the upper bound in Bourgain's Embedding Theorem is best possible up to constant.

Proof. By Lemma 16, for every n there exists a linear code C in $\mathbb{F}_2^n$ with $\dim C \geq \frac{n}{4}$ and $d(C) \geq \alpha n$. Let $X_n = \mathbb{F}_2^n / C^\perp$ with the quotient metric ρ. By Lemma 14, $|X_n| = 2^{n - \dim C^\perp} = 2^{\dim C} \geq 2^{n/4} \to \infty$. By Proposition 2, a lower bound on $c_1(X_n)$ is given by the Poincaré ratio corresponding to the inequality in Theorem 18. Thus

$$\begin{aligned} c_1(X_n) &\geq \frac{\int_{X_n \times X_n} \rho(u,v)\, d\tilde{\mu}(u)\, d\tilde{\mu}(v)}{\frac{1}{d(C)} \sum_{j=1}^n \int_{X_n} \frac{\rho(u + q(e_j), u)}{2}\, d\tilde{\mu}(u)} \\ &= \frac{\int_{\mathbb{F}_2^n \times \mathbb{F}_2^n} \rho(q(x), q(y))\, d\mu(x)\, d\mu(y)}{\frac{1}{2d(C)} \sum_{j=1}^n \int_{\mathbb{F}_2^n} \rho(q(x + e_j), q(x))\, d\mu(x)}. \end{aligned}$$

It's clear that the denominator $\leq \frac{n}{2d(C)} \leq \frac{n}{2\alpha n} = \frac{1}{2\alpha}$. By Lemma 19, for each $x \in \mathbb{F}_2^n$, $\int_{\mathbb{F}_2^n} \rho(q(x), q(y))\, d\mu(y) \geq \frac{\beta n}{2}$. Hence the numerator is at least $\frac{\beta n}{2}$. Thus $c_1(X_n) \geq \frac{\beta n}{2} / \frac{1}{2\alpha} = \alpha\beta n \geq \alpha\beta \log_2 |X_n|$. □

5 Dimension Reduction

Theorem 5.1 (Johnson-Lindenstrauss Lemma)**.** There exists a constant $C > 0$ such that $\forall k, n \in \mathbb{N}, \forall \epsilon \in (0, 1)$, if $k \geq C\epsilon^{-2} \log n$ then any n-element subset of ℓ_2 embeds into ℓ_2^k with distortion at most $\frac{1+\epsilon}{1-\epsilon}$.

Remark. In the 90's there was a sudden explosion of citation for this result, because the computer scientists realised there are many applications in compress sensing etc. For applications, see Matousek's lecture notes.

Idea. We will take a random linear map $T\colon \ell_2^n \to \ell_2^k$ and show that for each $x \in \ell_2^n$, we have $(1-\epsilon)\|x\|_2 \leq \|Tx\|_2 \leq (1+\epsilon)\|x\|_2$ with high probability. It follows that, given $x_1, \ldots, x_n \in \ell_2^n$, we have

$$(1-\epsilon)\|x_i - x_j\|_2 \leq \|Tx_i - Tx_j\|_2 \leq (1+\epsilon)\|x_i - x_j\|_2$$

with positive probability. In particular, there exists a suitable map of $\{x_1, \ldots, x_n\}$ to ℓ_2^k.

Lemma 5.2. Let $k, n \in \mathbb{N}, \epsilon(0, 1)$. Define $T\colon \ell_2^n \to \ell_2^k$ by the $k \times n$ matrix $(\frac{1}{\sqrt{k}} Z_{ij})_{ij}$ where the Z_{ij} $(1 \leq i \leq k, 1 \leq j \leq n)$ are iid random variables with $Z_{ij} \sim N(0, 1)$. Then there exists a constant $c > 0$ (independent of k, ϵ) such that for each $x \in \ell_2^n$, we have

$$\mathbb{P}\Big((1-\epsilon)\|x\|_2 \leq \|Tx\|_2 \leq (1+\epsilon)\|x\|_2\Big) \geq 1 - 2e^{-ck\epsilon^2}.$$

Proof of Theorem 1. We choose $C > 0$ sufficiently large so that if $k, n \in \mathbb{N}, \epsilon \in (0, 1)$ satisfy $k \geq C\epsilon^{-2} \log n$, then $1 - 2e^{-ck\epsilon^2} \geq 1 - \frac{1}{n^2}$. Clearly, C depends only on c. Now let $T\colon \ell_2^n \to \ell_2^k$ be as in Lemma 2. Then for each $x \in \ell_2^n$,

$$\mathbb{P}\Big((1-\epsilon)\|x\|_2 \leq \|Tx\|_2 \leq (1+\epsilon)\|x\|_2\Big) \geq 1 - \frac{1}{n^2}.$$

So given $x_1, \ldots, x_n \in \ell_2$, WLOG $x_1, \ldots, x_n \in \ell_2^n$ and

$$\mathbb{P}\Big(\forall i, j \quad (1-\epsilon)\|x_i - x_j\|_2 \leq \|Tx_i - Tx_j\|_2 \leq (1+\epsilon)\|x_i - x_j\|_2\Big) \geq 1 - \binom{n}{2}\frac{1}{n^2} > 0.$$

So there exists a linear map T that has $\frac{1+\epsilon}{1-\epsilon}$-distortion on $\{x_1, \ldots, x_n\}$. $\square$

Recall that if $Z \sim N(0, 1)$ then Z has probability density function (pdf) $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. If $Z_1, \ldots, Z_n$ are iid $\sim N(0, 1)$ and $x \in \ell_2^n$ with $\|x\| = \sqrt{\sum_{i=1}^n x_i^2} = 1$, then $\sum_{i=1}^n x_i Z_i \sim N(0, 1)$.

Lemma 5.3 (Tail Estimates)**.** Let X be a random variable with $\mathbb{E}X = 0$. Assume that for some $C > 0, u_0 > 0$ we have $\mathbb{E}e^{uX} \leq e^{Cu^2}$ for $0 \leq u \leq u_0$. Then $\mathbb{P}(X > t) \leq e^{-t^2/4C}$ for $0 \leq t \leq 2Cu_0$.

Proof. For any $u \geq 0$,

$$\begin{aligned} \mathbb{P}(X > t) = \mathbb{P}(e^{uX} > e^{ut}) &\leq e^{-ut}\mathbb{E}e^{uX} && \text{(Markov's inequality)} \\ &\leq e^{-ut + Cu^2} && \text{(provided } 0 \leq u \leq u_0\text{)} \end{aligned}$$

If $0 \leq t \leq 2Cu_0$, then we can take $u = t/2C$ to obtain

$$\mathbb{P}(X > t) \leq e^{-\frac{t^2}{2C} + \frac{t^2}{4C}} = e^{-\frac{t^2}{4C}}.$$

□

Lemma 5.4. Assume $Z \sim N(0,1)$. Then there exists absolute constant $C, u_0 > 0$ such that $\mathbb{E}e^{u(Z^2-1)} \leq e^{Cu^2}$ and $\mathbb{E}e^{u(1-Z^2)} \leq e^{Cu^2}$ for $0 \leq u \leq u_0$.

Proof. This is straightforward computation.

$$\begin{aligned}
\mathbb{E}e^{u(1-Z^2)} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{u(1-x^2)} e^{-x^2/2}\, dx \\
&= e^u \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(2u+1)x^2}\, dx \\
&= \frac{e^u}{\sqrt{2u+1}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2}\, dy && (\text{put } y = \sqrt{2u+1}x) \\
&= \frac{e^u}{\sqrt{2u+1}} \\
&= e^{u - \frac{1}{2}\log(2u+1)} \\
&= e^{u^2 + O(u^3)}
\end{aligned}$$

using $\log(1+x) = -\sum_{n=1}^\infty \frac{(-x)^n}{n}$. A similar computation shows $\mathbb{E}e^{u(Z^2-1)} \leq e^{u^2+O(u^3)}$. □

Proof of Lemma 2. Fix $x \in \ell_2^n$. WLOG assume $\|x\|_2 = 1$. Then

$$(Tx)_i = \frac{1}{\sqrt{k}} \sum_{j=1}^n x_j Z_{ij}, \quad 1 \leq i \leq k.$$

Let $Z_i = \sum_{j=1}^n x_j Z_{ij}$. Then $Z_1, \ldots, Z_n$ are iid with $Z_i \sim N(0,1)$. Then

$$\mathbb{E}\|Tx\|^2 = \sum \mathbb{E}|(Tx)_i|^2 = \frac{1}{k}\sum_{i=1}^k \mathbb{E}(Z_i^2) = 1.$$

Let $W = \frac{1}{\sqrt{k}} \sum_{i=1}^k (Z_i^2 - 1)$. Then $\mathbb{E}W = 0$ and $\operatorname{var} W = 1$. Fix C, u_0 as given by Lemma 4 and WLOG $2cu_0 \geq 1$. Then

$$\begin{aligned}
\mathbb{E}e^{uW} &= \prod_{i=1}^k e^{\frac{u}{\sqrt{k}}(Z_i^2-1)} && (\text{by independence}) \\
&\leq \prod_{i=1}^k e^{Cu^2/k} && (\text{Lemma 4}) \\
&= e^{Cu^2} && (\text{if } 0 \leq u \leq \sqrt{k}u_0).
\end{aligned}$$

Similarly $\mathbb{E}(e^{-uW}) = \prod_{i=1}^{k} e^{u/sqrtk(1-Z_i^2)} \leq e^{Cu^2}$. So $\mathbb{P}(W > t) \leq e^{-t^2/4C}$, $\mathbb{P}(W < -t) \leq e^{-t^2/4C}$ for $0 \leq t \leq 2Cu_0\sqrt{k}$ (note $2Cu_0 \geq 1$). So

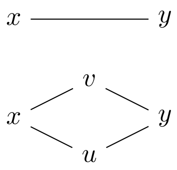
$$\begin{aligned}\mathbb{P}\Big((1-\epsilon)\|x\|_2 \leq \|Tx\|_2 \leq &(1+\epsilon)\|x\|_2\Big) \\ &= \mathbb{P}\Big((1-\epsilon)^2 \leq \|Tx\|_2^2 \leq (1+\epsilon)^2\Big) \\ &\geq \mathbb{P}\Big(1-\epsilon \leq \frac{1}{k}\sum_{i=1}^{k} Z_i^2 \leq 1+\epsilon\Big) \\ &= \mathbb{P}\Big(1-\epsilon \leq \frac{1}{\sqrt{k}} W + 1 \leq 1+\epsilon\Big) \\ &= \mathbb{P}(-\epsilon\sqrt{k} \leq W \leq \epsilon\sqrt{k}) \\ &\geq 1 - 2e^{-\epsilon^2 k/4C}.\end{aligned}$$

□

Aim. Our aim is to prove that dimension reduction as in JL Lemma does not work in ℓ_1.

Theorem 5.5. For all $n \in \mathbb{N}$ there exists a subset X of ℓ_1 of size $|X| = N \geq n$ such that if X embeds into ℓ_1^k with distortion $\leq D$, then $k \geq n^{\frac{1}{32D^2}}$.

We introduce the *diamond graphs* D_n, $n = 0, 1, 2, \dots$: D_0 consists of 2 vertices joined by an edge. D_{n+1} is obtained from D_n by replacing every edge xy in D_n with new vertices u, v and edges xv, vy, xu, uy. Note $D_0 = K_2, D_1 = C_4$.



Let $E_n = E(D_n), V_n = V(D_n)$. Then $|E_n| = 4^n, |V_n| = 2 + 2(1 + 4 + \cdots + 4^{n-1}) = \frac{2}{3}(4^n + 2)$. Observe that $|V_n| \leq 4^n$ for all $n \geq 1$.

Let $d_n = d_{D_n}$. For every $n \geq m \geq 0, \forall x, y \in D_m$, $d_n(x, y) = 2^{n-m} d_m(x, y)$.

We define sets A_n for $n \geq 1$ of "non-edges" as follows: For $n \geq 1$, D_n consists of copies of $D_1 = C_4$ of the form $xyuv$ where $xy \in E_{n-1}$ and $u, v \in V_n \setminus V_{n-1}$. Let A_n consist of all pairs $\{u, v\}$.

Let's label the vertices as follows. $D_0 = \ell r$ for left and right, $D_1 = \ell brt$ where b for bottom and t for top. Write $D_n(\ell r)$ for D_n. $D_{n+1}(\ell r)$ consists of 4 copies of D_n: $D_n(t\ell), D_n(tr), D_n(b\ell), D_n(br)$. If e, f are two of the edges $t\ell, tr, b\ell, br$, then $V(D_n(e)) \cap V(D_n(f)) = e \cap f$.

Remark. $d_n(\ell, r) = 2^n$ for all $n \geq 0$, and $d_n(t, b) = 2^n$ for all $n \geq 1$. For every $x \in D_n$, $d_n(\ell, x) + d_n(x, r) = 2^n$.

Lemma 5.6. For all $n \geq 0$, D_n embeds into $\ell_1^{2^n}$ with distortion ≤ 2.

Proof. Let $f_0 \colon D_0 \to H_k \subset \ell_1^k$ be such that $f_0(\ell), f_0(r)$ are neighbours in H_k. So f_0 is isometric (e.g. $k = 1 = 2^0$, $f_0(\ell) = (0), f_0(r) = (1)$). Assume $f_n \colon D_n \to$

$H_{k2^n} \subset \ell^{k2^n}$ has been defined. Then we define $f_{n+1}: D_{n+1} \to H_{k2^{n+1}} \subset \ell^{k2^{n+1}}$ as follows: let $x \in D_n$ we let $f_{n+1}(x) = (f_n(x), f_n(x))$. If $xy \in E_n$ and u, v are the corresponding new vertices in D_{n+1}, we let $f_{n+1}(u) = (f_n(x), f_n(y))$, $f_{n+1}(v) = (f_n(y), f_n(x))$.

Observe that for $x, y \in D_n$, $\|f_{n+1}(x) - f_{n+1}(y)\|_1 = 2\|f_n(x) - f_n(y)\|_1$. So $\forall n \geq m \geq 0, \forall x, y \in D_m$, $\|f_n(x) - f_n(y)\|_1 = 2^{n-m}\|f_m(x) - f_m(y)\|_1$.

First show that $\forall n \geq 0, \forall xy \in E_n$, $\|f_n(x) - f_n(y)\|_1 = d_n(x, y) = 1$. Proof by induction on n: $n = 0$ (and $n = 1$) is clear. Now assume $n \geq 1$. An edge in D_n is of the form xu, where $\exists xy \in E_{n-1}$ and u, v are the corresponding new vertices in D_n. Now $\|f_n(x) - f_n(y)\| = \|(f_{n-1}(x), f_{n-1}(x)) - (f_{n-1}(x), f_{n-1}(y))\|_1 = \|f_{n-1}(x) - f_{n-1}(y)\|_1 = 1$ by induction hypothesis. It follows that f_n is 1-Lipschitz for all $n \geq 0$. To see this, given $x, y \in D_n$, there exists a path $x = x_0, x_1, x_2, \ldots, x_m = y$ in D_n with $m = d_n(x, y)$. Then $\|f_n(x) - f_n(y)\|_1 \leq \sum_{i=1}^m \|f_n(x_i) - f_n(x_{i-1})\|_1 = m = d_n(x, y)$.

Claim. $\forall n \geq 0, \forall x, y \in D_n$, $\|f_n(x) - f_n(y)\|_1 \geq \frac{1}{2} d_n(x, y)$.

Note that $\forall n \geq m \geq 0$, if $xy \in E_m$, then $\|f_n(x) - f_n(y)\|_1 = 2^{n-m}\|f_m(x) - f_m(y)\|_1 = 2^{n-m} = 2^{n-m} d_m(x, y) = d_n(x, y)$. In fact, it is enough if $\|f_m(x) - f_m(y)\|_1 = d_m(x, y)$.

This claim is proved by induction on n. Note that f_0, f_1 are isometric. Assume $n \geq 2$ and the claim holds for $n - 1$. Fix $x, y \in D_n$. Recall that D_n consists of 4 copies of D_{n-1}. We have 3 cases.

Case 1: x, y in the same copy, WLOG $x, y \in D_{n-1}(t\ell)$. Define $g_0: D_0(t\ell) \to H_{2k}$, $g_0(u) = f_1(u)$. Then define $g_m: D_m \to H_{2^m k}$ inductively starting with g_0 in the same way as f_m is defined from f_0. Then by easy induction, $g_{n-1} = f_n|_{D_{n-1}(t\ell)}$. By induction hypothesis, $\|f_n(x) - f_n(y)\|_1 = \|g_{n-1}(x) - g_{n-1}(y)\|_1 \geq \frac{1}{2} d_{D_{n-1}(t\ell)}(x, y) \geq \frac{1}{2} d_{D_n}(x, y)$. [In fact, the last inequality is an equality, because the four copies of D_{n-1} only meet at ℓ, b, r or t.]

Case 2: x, y are in neighbouring copies, WLOG $x \in D_{n-1}(t\ell), y \in D_{n-1}(tr)$. Now $\|f_n(x) - f_n(y)\| \geq \|f_n(\ell) - f_n(r)\|_1 - \|f_n(\ell) - f_n(x)\|_1 - \|f_n(y) - f_n(r)\|_1 = 2^{n-1}\|f_1(\ell) - f_1(r)\| - d_n(x, \ell) - d_n(y, r) = 2^n - d_n(x, \ell) - d_n(y, r) = \left(2^{n-1} - d_{D_{n-1}(t\ell)}(x, \ell)\right) + \left(2^{n-1} - d_{D_{n-1}(tr)}(y, r)\right) = d_n(x, t) + d_n(t, y) = d_n(x, y)$.

Case 3: x, y are in opposite copies, WLOG $x \in D_{n-1}(t\ell), y \in D_{n-1}(br)$. Then

$$d_n(x, y) = \left(d_n(x, \ell) + 2^{n-1} + d_n(b, y)\right) \wedge \left(d_n(x, t) + 2^{n-1} + d_n(r, y)\right) \leq 2^n,$$

since $d_n(x, \ell) + d_n(b, y) + d_n(x, t) + d_n(r, y) = 2^n$. Assume WLOG $d_n(x, t) + d_n(y, b) \leq d_n(x, \ell) + d_n(y, r)$. So $d_n(x, t) + d_n(y, b) \leq 2^{n-1}$. Then by the triangle inequality and the fact that f_n is 1-Lipschitz,

$$\begin{aligned} \|f_n(x) - f_n(y)\|_1 &\geq \|f_n(t) - f_n(b)\|_1 - \|f_n(x) - f_n(t)\|_1 - \|f_n(y) - f_n(b)\|_1 \\ &\geq 2^n - d_n(x, t) - d_n(y, b) \geq 2^{n-1} \geq \frac{1}{2} d_n(x, y). \end{aligned}$$

□

Recall that for all $x_1, x_2, x_3, x_4 \in \ell_2$ we have

$$\begin{aligned}\|x_1 - x_3\|_2^2 + \|x_2 - x_4\|_2^2 \leq \|x_1 - x_2\|_2^2 + \|x_2 - x_3\|_2^2 \\ + \|x_3 - x_4\|_2^2 + \|x_4 - x_1\|_2^2,\end{aligned}$$

also called the *Short Diagonal Lemma*.

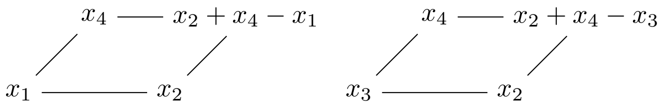
Lemma 5.7 (Short diagonal Lemma in L_p)**.** Let $1 < p < 2$. Then $\forall x_1, x_2, x_3, x_4 \in L_p$, we have

$$\begin{aligned}\|x_1 - x_3\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \leq \|x_1 - x_2\|_p^2 + \|x_2 - x_3\|_p^2 \\ + \|x_3 - x_4\|_p^2 + \|x_4 - x_1\|_p^2,\end{aligned}$$

Proof. WLOG $x_1, x_2, x_3, x_4 \in \ell_p^k$ for some k ($k = 6$ will do by Theorem 2.7). Lemma 7 can be deduced from the following:

$$\|x\|_p^2 + (p-1)\|y\|_p^2 \leq \frac{\|x+y\|_p^2 + \|x-y\|_p^2}{2} \qquad \forall x, y \in \ell_p^k. \qquad (*)$$

To see this, consider two parallelograms:



For the first parallelogram, set $x = x_2 + x_4 - 2x_1, y = x_4 - x_2$. For the second parallelogram, set $x = x_2 + x_4 - 2x_3, y = x_4 - x_2$. Apply $(*)$ for both parallelograms:

$$\begin{aligned}\|x_2 + x_4 - 2x_1\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \leq 2\|x_4 - x_1\|_p^2 + 2\|x_2 - x_1\|_p^2, \\ \|x_2 + x_4 - 2x_3\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \leq 2\|x_4 - x_3\|_p^2 + 2\|x_2 - x_3\|_p^2.\end{aligned}$$

We take average of these 2 inequalities and use convexity of $z \mapsto \|z\|_p^2$ to get

$$\begin{aligned}
&\|x_1 - x_3\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \\
&\quad = \left\|\frac{x_2 + x_4 - 2x_3}{2} + \frac{2x_1 - x_2 - x_4}{2}\right\|_p^2 + (p-1)\|x_2 - x_4\|_p^2 \\
&\quad \leq \frac{\|x_2 + x_4 - 2x_3\|_p^2 + \|x_2 + x_4 - 2x_1\|_p^2}{2} + (p-1)\|x_2 - x_4\|_p^2 \\
&\quad \leq \|x_1 - x_2\|_p^2 + \|x_2 - x_3\|_p^2 \\
&\qquad\qquad + \|x_3 - x_4\|_p^2 + \|x_4 - x_1\|_p^2,
\end{aligned}$$

as required.

To prove $(*)$, use the fact that for $a, b \geq 0$, $\left(\frac{a^q + b^q}{2}\right)^{1/q}$ is increasing in $q \in [1, \infty)$. So $(*)$ follows from

$$\|x\|_p^2 + (p-1)\|y\|_p^2 \leq \left(\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2}\right)^{2/p}.$$

Define

$$\begin{aligned} L(t) &= \|x\|_p^2 + (p-1)\|y\|_p^2 t^2, \\ R(t) &= \left(\frac{\|x+ty\|_p^p + \|x-ty\|_p^p}{2}\right)^{2/p} = H(t)^{2/p}, \\ H(t) &= \frac{1}{2}\sum_{i=1}^k \left(|x_i + ty_i|^p + |x_i - ty_i|^p\right), \qquad t \in \mathbb{R}. \end{aligned}$$

We need that $L(1) \le R(1)$. We have $L(0) = R(0) = \|x\|_p^2$. From now we assume $x \neq 0, y \neq 0$. Next we differentiate.

$$\begin{aligned} L'(t) &= 2(p-1)\|y\|_p^2 t \\ R'(t) &= \frac{2}{p} H(t)^{\frac{2}{p}-1} H'(t) \\ H'(t) &= \frac{p}{2}\sum_{i=1}^k \big(|x_i + ty_i|^{p-1}\operatorname{sgn}(x_i + ty_i) y_i - |x_i - ty_i|^{p-1}\operatorname{sgn}(x_i - ty_i) y_i\big). \end{aligned}$$

Note that $L'(0) = R'(0) = 0$. Differentiate again:

$$L''(t) = 2(p-1)\|y\|_p^2.$$

Let $I = [k] \setminus \{i \in [k] : x_i = y_i = 0\}$, where $[k] = \{1, \ldots, k\}$. Note $I \neq \emptyset$ as $x, y \neq 0$. For $i \in I$, there is ≤ 1 value of t such that $x_i + ty_i = 0$. So there exists dissection $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0,1]$ such that $x_i + ty_i \neq 0, \forall i \in I, \forall t \in \bigcup_{j=1}^m (t_{j-1}, t_j)$. For such t, we have

$$\begin{aligned} R''(t) &= \frac{2}{p}\left(\frac{2}{p} - 1\right) H(t)^{\frac{2}{p}-2}(H'(t))^2 + \frac{2}{p} H(t)^{\frac{2}{p}-1} H''(t) \\ &\ge \frac{2}{p} H(t)^{\frac{2}{p}-1} H''(t) \\ &= \frac{2}{p} H(t)^{\frac{2}{p}-1} \frac{p}{2}(p-1)\sum_{i\in I}\big(|x_i + ty_i|^{p-2} y_i^2 + |x_i - ty_i|^{p-2} y_i^2\big). \end{aligned}$$

We now use *reverse Hölder's inequality*: suppose $0 < r < 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, so $s = \frac{r}{r-1} < 0$. Given $a_i, b_i \in \mathbb{R}$, $b_i \neq 0$, we have

$$\begin{aligned} \left(\sum_{i\in I} |a_i|^r\right)^{1/r} &= \left(\sum_{i\in I} |a_i b_i|^r |b_i|^{-r}\right)^{1/r} \qquad \left(\text{take } p = \frac{1}{r}, q = \frac{1}{1-r}\right) \\ &\le \left(\sum_{i\in I} |a_i b_i|\right)\left(\sum_{i\in I} |b_i|^s\right)^{-1/s}, \end{aligned}$$

so

$$\left(\sum_{i\in I} |a_i|^r\right)^{1/r}\left(\sum_{i\in I} |b_i|^s\right)^{1/s} \le \sum_{i\in I} |a_i b_i|.$$

Apply this with $b_i = |x_i \pm ty_i|^{p-2}$, $a_i = y_i^2$, $r = \frac{p}{2}$, $s = \frac{p}{p-2}$, we have

$$\begin{aligned}
R''(t) &\geq H(t)^{\frac{2}{p}-1}(p-1)\left(\sum_{i\in I}|y_i|^p\right)^{2/p}\left(\left(\sum_{i\in I}|x_i+ty_i|^p\right)^{\frac{p-2}{p}}+\left(\sum_{i\in I}|x_i-ty_i|^p\right)^{p}\right. \\
&\geq H(t)^{\frac{2}{p}-1}(p-1)\|y\|_p^2\cdot 2\left(\frac{\|x+ty\|_p^{p-2}+\|x-ty\|_p^{p-2}}{2}\right) \\
&\geq H(t)^{\frac{2}{p}-1}(p-1)2\|y\|_p^2\left(\frac{\|x+ty\|_p^{p}+\|x-ty\|_p^{p}}{2}\right)^{\frac{p-2}{p}} \qquad (r\mapsto r^{\frac{p-2}{p}}\text{ convex}) \\
&= 2(p-1)\|y\|_p^2 = L''(t).
\end{aligned}$$

So for each $1 \leq j \leq m$, $(R-L)'' \geq 0$ on (t_{j-1}, t_j), so $(R-L)'$ is increasing on $[t_{j-1}, t_j]$. So $(R-L)'$ is increasing on $[0,1]$ and hence $(R-L)' \geq 0$ on $[0,1]$. So $R-L$ is increasing on $[0,1]$ and hence $R(1)-L(1) \geq 0$. □

Corollary 5.8. For $1 < p < 2, n \in \mathbb{N}$, $c_p(D_2) \geq \sqrt{1+(p-1)n}$.

Proof. D_n consists of copies of $D_1 = xuyv$, where $xy \in E_{n-1}, uv \in V_n \setminus V_{n-1}$. Apply Lemma 7 for a function $f\colon D_n \to L_p$:

$$\begin{aligned}
\|f(x)-f(u)\|_p^2 + \|f(u)-f(y)\|_p^2 + \|f(y)-f(v)\|_p^2 + \|f(v)-f(x)\|_p^2& \\
\geq \|f(x)-f(y)\|_p^2 + (p-1)\|f(u)-f(v)\|_p^2&.
\end{aligned}$$

Sum over all copies of D_1 in D_n:

$$\begin{aligned}
\sum_{xy\in E_n}\|f(x)-f(y)\|_p^2 &\geq \sum_{xy\in E_{n-1}}\|f(x)-f(y)\|_p^2 + (p-1)\sum_{xy\in A_n}\|f(x)-f(y)\|_p^2 \\
&\geq \cdots \\
&\geq \|f(\ell)-f(r)\|_p^2 + (p-1)\sum_{xy\in A_1\cup\cdots\cup A_n}\|f(x)-f(y)\|_p^2.
\end{aligned}$$

We bound $c_p(D_n)$ from below using the corresponding Poincaré ratio. For $xy \in A_k$, $d_n(x,y) = 2^{n-k}d_k(x,y) = 2^{n-k+1}$ and $|A_k| = 4^{k-1}$. So $d_n(\ell, r)^2 + (p-1)\sum_{k=1}^n 4^{k-1}4^{n-k+1} = 4^n(1+(p-1)n)$. So $c_p(D_n) \geq \left(\frac{4^n(1+(p-1)n)}{4^n}\right)^{1/2} = \sqrt{1+(p-1)n}$. □

Lemma 5.9. Given $k \geq 2$, the identity $i_p\colon \ell_1^k \to \ell_p^k$ where $p = 1 + \frac{1}{\log_2 k}$ has distortion at most 2.

Proof. For $x = (x_i)_{i=1}^k \in \mathbb{R}^k$, by Hölder, $\|x\|_p \leq \|x\|_1 = \sum_{i=1}^k |x_i| \leq k^{1-1/p}\|x\|_p$. Now $k^{1-1/p} = k^{\frac{1/\log_2 k}{1+1/\log_2 k}} = k^{\frac{1}{\log_2 k+1}} = 2^{\frac{\log_2 k}{\log_2 k+1}} \leq 2$. □

Proof of Theorem 5. Let $n \in \mathbb{N}$. By Theorem 6, there exists an embedding $f\colon D_n \to \ell_1$ of distortion at most 2. Set $X = f(D_n)$. So $|X| = |D_n| \leq 4^n$. Assume $g\colon X \to \ell_1^k$ has distortion at most D. Then $i_p g f\colon X \to \ell_p^k$, $p = 1 + \frac{1}{\log_2 k}$ has distortion $\leq 4D$ (Lemma 9). By Corollary 8, $4D \geq \sqrt{1+(p-1)n}$, and $16D^2 \geq \frac{n}{\log_2 k} \geq \frac{\log_2|X|}{2\log_2 k}$. So $\log_2 k \geq \frac{\log_2|X|}{32D^2}$ and hence $k \geq |X|^{\frac{1}{32D^2}}$. □

6 Ribe Programme

Definition. Given Banach spaces X, Y, we say X is *finitely representable* in Y if $\forall E \subset X, \dim E < \infty$, $\forall \lambda > 1$, $\exists F \subset Y$ such that $d(E,F) < \lambda$, i.e. there exists a linear bijection $T\colon E \to F$ such that $\|T\|\|T^{-1}\| < \lambda$.

Example.

(i) Every X is finitely representable in c_0.

(ii) ℓ_2 is finitely representable in every ∞-dimensional X [Dvoretzky].

Definition. X is *crudely finitely representable* in Y if $\exists \lambda > 1$, $\forall E \subset X$, $\dim E < \infty$, $\exists F \subset Y$, s.t. $d(E,F) < \lambda$.

Definition. A *local property* (or *local isomorphic property*) of a Banach space is one that depends only on its finite-dimensional subspaces.

Definition. For $1 \leq p \leq 2$, we say X has *type* p if $\exists C > 0$, $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n \in X$, $\mathbb{E}\|\sum_{i=1}^n \epsilon_i x_i\| \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$. Here, $\epsilon_1, \ldots, \epsilon_n$ are $\{\pm 1\}$-valued independent $\mathtt{Rademacher}(\frac{1}{2})$ random variables.

For $2 \leq q \leq \infty$, we say X has *cotype* q if $\exists C > 0$, $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n \in X$, $\mathbb{E}\|\sum_{i=1}^n \epsilon_i x_i\| \geq \frac{1}{C}\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q}$. For $q = \infty$, RHS $= \frac{1}{C}\max_{1\leq i \leq n}\|x_i\|$.

Example. Every X has type 1, cotype ∞; ℓ_2 has type 2 and cotype 2 with $C = 1$.

If X is crudely finitely representable in Y and Y has some local property, then so does X.

Theorem 6.1 (Ribe's Theorem)**.** If Banach spaces X, Y are uniformly homeomorphic then X is crudely finitely representable in Y and vice versa.

Proof. Omitted. □

Remark. Local properties depend only on the metric structure of the Banach space, not the linear structure.

Aim. Aim for the Ribe programme:

(i) Find metric characterisations of local properties of Banach spaces.

(ii) Find metric analogues of local properties of Banach spaces.

Our aim is to find a metric characterisation of *super-reflexivity*.

Definition. Recall that given a Banach space X, there is an isometric isomorphism $X \longrightarrow X^{**}$ $x \longmapsto \hat{x}$, where $\hat{x}(f) = f(x)$. Easy to check $\hat{x} \in X^{**}$ and $\|\hat{x}\| \leq \|x\|$. By Hahn-Banach, we have $\|\hat{x}\| = \|x\|$. It's then clear that $x \longmapsto \hat{x}$ is linear. So the image of X in X^{**} is a closed subspace of X^{**}, which we will always identify with X. Say X is reflexive if $X = X^{**}$.

Warning. There exists Banach space J such that J is isometrically isomorphic to J^{**} but J^{**}/J has dimension 1.

Definition. We say X is *super-reflexive* if every Y finitely representable in X is reflexive. So super-reflexive $\implies$ reflexive.

Example. Let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_1^n\right)_{\ell_2} = \{(x_n) : x_n \in \ell_1^n \,\forall n, \sum \|x_n\|^2 < \infty\}$. X is reflexive, but ℓ_1 is finitely representable in X (see example sheet), so X is not super-reflexive.

We recall the following for a Banach space X:

(i) The *weak topology* on X is defined as follows: $U \subset X$ is *w-open* if $\forall x \in U$, $\exists n \in \mathbb{N}$, $\exists f_1, \dots, f_n \in X^*$, $\exists \epsilon > 0$ such that $\{y : |f_i(y - x)| < \epsilon, \forall i\} \subset U$. Note $|f_i(y-x)| < \epsilon$ can be written as $f_i(x) - \epsilon < f_i(y) < f_i(x) + \epsilon$. So this is a cylindrical set with finite codimension. This is the weakest topology on X for which every $f \in X^*$ is continuous.

(ii) A convex subset C of X is $\|\cdot\|$-closed $\iff$ w-closed.

Proof. ($\Leftarrow$) is clear. ($\Rightarrow$) if $x \notin C$, then by Hahn-Banach separation ($\{x\}$ compact convex, C closed convex), there exists $f \in X^*$ such that $\sup_C f < f(x)$. So $\{y : f(y) > \sup_C f\}$ is a weak neighbourhood of x disjoint from C. □

(iii) The *w^*-topology* on X^* is defined as follows: $U \subset X^*$ is *w^*-open* $\iff$ $\forall f \in U$, $\exists n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, $\epsilon > 0$ such that $\{g \in X^* : |(g - f)(x_i)| < \epsilon, \forall i\} \subset U$. This is the weakest topology on X^* for which every $x \in X \subset X^{**}$ is continuous. So w^*-topology $\subset$ w-topology on X^*.

(iv) **Banach-Alaoglu Theorem**: $B_{X^*} = \{f \in X^* : \|f\| \le 1\}$ is w^*-compact.

Proof. Define

$$(B_{X^*}, w^*) \xrightarrow{\varphi} \prod_{x\in X} \{\lambda \in \mathbb{R} : |\lambda| \le \|x\|\} \ ,$$

with $\varphi(f) = (f(x))_{x\in X}$ where the codomain is equipped with the product topology, which is compact by Tychonov. It's clear that φ is a homeomorphism of B_{X^*} onto $\varphi(B_{X^*})$. Then $\varphi(B_{X^*}) = \bigcap_{x,y\in X, a,b\in\mathbb{R}} \{(\lambda_x)_{x\in X} : \lambda_{ax+by} - a\lambda_x - b\lambda_y = 0\}$, which is closed, hence compact. □

(v) **Goldstine's Theorem**: $\overline{B_X}^{w^*} = B_{X^{**}}$ in X^{**}.

(vi) X is reflexive $\iff$ (B_X, w) is compact.

Proof. ($\Rightarrow$): We have $X = X^{**}$, so $(X, w) = (X^{**}, w^*)$ so $(B_X, w) = (B_{X^{**}}, w^*)$ which is compact by Banach-Alaoglu.

($\Leftarrow$): The restriction of the w^*-topology of X^{**} to X is the w-topology. So B_X is w^*-compact in X^{**}. So B_X is w^*-closed and hence $B_{X^{**}} = \overline{B_X}^{w^*} = B_X$ and hence $X^{**} = X$. □

Lemma 6.2 (Local reflexivity)**.** Let X be a Banach space, $E \subset X^*$ with $\dim E < \infty$ and let $\varphi \in X^{**}$ and let $M > \|\varphi\|$. Then $\exists x \in X$ such that $\|x\| < M$ and $\hat{x}|_E = \varphi|E$.

Remark. We can now prove Goldstine: $\overline{B_X}^{w^*} = B_{X^{**}}$. Since $B_X \subset B_{X^{**}}$ and $B_{X^{**}}$ is w^*-closed, it follows that $\overline{B_X}^{w^*} \subset B_{X^{**}}$. Fix $\psi \in B_{X^{**}}$ and a w^*-neighbourhood U of ψ. Then $\exists n \in \mathbb{N}, f_1, \ldots, f_n \in X^*, \exists \epsilon > 0$ such that $\{\chi \in X^{**} : |(\chi - \psi)(f_i)| < \epsilon, \forall i\} \subset U$. Fix $\delta > 0$ to be determined. By Lemma 2, $\exists x \in X, \|x\| < 1 + \delta$, and $f_i(x) = \psi(f_i)$ for all i. If $\|x\| \leq 1$, then $x \in B_X \cap U$, so done. Assume $\exists x\| > 1$. Then

$$\left|\frac{\hat{x}}{\|x\|}(f_i) - \psi(f_i)\right| = \left|\frac{f_i(x)}{\|x\|} - f_i(x)\right| = \frac{|f_i(x)|}{\|x\|}\,|1 - \|x\|| \leq \delta\|f_i\|, \quad \forall i.$$

We can choose $\delta > 0$ such that $\delta\|f_i\| < \epsilon$ for all i, and then $\frac{x}{\|x\|} \in B_X \cap U$.

Proof of Lemma 2. Fix a basis $f_1, \ldots, f_n$ of E. Define $T\colon X \longrightarrow \mathbb{R}^n$ by $Tx = (f_i(x))_{i=1}^n$ and let $C = \{Tx : \|x\| < M\}$. We need $(\varphi(f_i))_{i=1}^n \in C$. Then we will be done. T is a bounded linear map and C is convex. We show that T is onto: if not, then there exists $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$ such that $\sum_{i=1}^n a_i f_i(x) = 0$ for all x, i.e., $\sum_{i=1}^n a_i f_i = 0$, but this is a contradiction. By the Open Mapping Theorem, C is an open set. Let's assume that $(\varphi(f_i))_{i=1}^n \notin C$. By Hahn-Banach separation, $\exists a = (a_1, \ldots, a_n) \neq 0$ such that $\sum_{i=1}^n a_i f_i(x) < \sum_{i=1}^n a_i \varphi(f_i)$ for all $x \in X, \|x\| < M$. Hence $\|\sum_{i=1}^n a_i f_i\| M \leq \varphi(\sum_{i=1}^n a_i f_i) \leq \|\varphi\| \|\sum_{i=1}^n a_i f_i\|$. Since $\sum_{i=1}^n a_i f_i \neq 0$, we get $M \leq \|\varphi\|$, a contradiction. ☐

Theorem 6.3. Let X be a Banach space. Then the following are equivalent:

(i) X is non-reflexive;

(ii) $\forall \theta \in (0,1)$, $\exists (x_i)_{i=1}^\infty$ in B_X, $(f_i)_{i=1}^\infty$ in B_{X^*}, such that

$$f_i(x_j) = \begin{cases} \theta & \text{if } i \leq j \\ 0 & \text{if } i > j; \end{cases}$$

(iii) $\exists \theta \in (0,1)$, the above holds;

(iv) $\forall \theta \in (0,1)$, $\exists (x_i)$ in B_X such that $\forall n \in \mathbb{N}$,

$$d(\operatorname{conv}\{x_1, \ldots, x_n\}, \operatorname{conv}\{x_{n+1}, x_{n+2}, \ldots\}) \geq \theta.$$

(v) $\exists \theta \in (0,1)$, such that the above holds.

Proof. (i) $\implies$ (ii): Since X is a proper closed subspace of X^{**}, $\exists T \in X^{***}$ such that $\|T\| = 1$, $T|_X = 0$ (by Hahn-Banach). Fix $\theta \in (0,1)$ and choose $\varphi \in X^{**}$, $\|\varphi\| < 1$, $T(\varphi) > \theta$. Let $\lambda = T(\varphi)$. Then $\theta < \lambda = T(\varphi) \leq \|T\|\|\varphi\| = \|\varphi\| < 1$, i.e. $\theta < \lambda < 1$.

Since $\|\varphi\| > \theta$, there exists $f_1 \in B_{X^*}$ such that $\varphi(f_1) = \theta$. Then $\theta = \varphi(f_1) \leq \|\varphi\|\|f_1\| < \|f_1\|$, and hence $\exists x_1 \in B_X$ such that $f_1(x_1) = \theta$.

Assume now that for some $n \geq 1$ we have found sequences $(x_i)_{i=1}^n$ in B_X and $(f_i)_{i=1}^n$ in B_{X^*} such that

$$f_i(x_j) = \begin{cases} \theta & \text{if } 1 \leq i \leq j \leq n \\ 0 & \text{if } 1 \leq j < i \leq n, \end{cases}$$

and $\varphi(f_i) = \theta$ for $1 \le i \le n$. Since $T(x_i) = 0$ for $1 \le i \le n$ and $T(\varphi) = \lambda$ and $\|T\| = 1 < \frac{\lambda}{\theta}$, by Lemma 2, $\exists g \in X^*$ such that $\|g\| < \frac{\lambda}{\theta}$ and $g(x_i) = 0$ for $1 \le i \le n$ and $\varphi(g) = \lambda$. Then $f_{n+1} = \frac{\theta}{\lambda} g \in B_{X^*}$ and $f_{n+1}(x_i) = 0$ for $1 \le i \le n$ and $\varphi(f_{n+1}) = \theta$. Since $\varphi(f_i) = \theta$ for $1 \le i \le n+1$ and $\|\varphi\| < 1$, so by Lemma 2, $\exists x_{n+1} \in B_X$ such that $f_i(x_{n+1}) = \theta$ for $1 \le i \le n+1$. Now continue inductively.

(ii) $\implies$ (iii) and (iv) $\implies$ (v) are clear.

We next show (ii) $\implies$ (iv) and (iii) $\implies$ (v). Fix $\theta \in (0,1)$. Assume $\exists (x_i)$ in B_X, (f_i) in B_{X^*} such that

$$f_i(x_j) = \begin{cases} \theta & \text{if } i \le j \\ 0 & \text{if } i > j. \end{cases}$$

Given $n \in \mathbb{N}$ and finite convex combinations $\sum_{i=1}^n t_i x_i$ and $\sum_{i=n+1}^\infty t_i x_i$, we have

$$\left\| \sum_{i=n+1}^\infty t_i x_i - \sum_{i=1}^n t_i x_i \right\| \ge \left| f_{n+1}\Big(\sum_{i=n+1}^\infty t_i x_i - \sum_{i=1}^n t_i x_i \Big) \right| = \sum_{i=n+1}^\infty \theta t_i = \theta.$$

Thus

$$d(\mathrm{conv}\{x_1, \dots, x_n\}, \mathrm{conv}\{x_{n+1}, x_{n+2}, \dots\}) \ge \theta.$$

Finally, we show (v) $\implies$ (i). Assume $\exists \theta \in (0,1)$ and (x_i) in B_X such that (v) holds. Assume for a contradiction that X is reflexive.

For $n \in \mathbb{N}$, let $C_n = \mathrm{conv}\{x_{n+1}, x_{n+2}, \dots\}$. $\overline{C_n}$ ($\|\cdot\|$-closure) is a $\|\cdot\|$-closed, convex subset of B_X. Hence $\overline{C_n}$ is a w-closed subset of B_X. Also $\overline{C_1} \supset \overline{C_2} \supset \overline{C_3} \supset \dots$ and $\overline{C_n} \ne \emptyset$ for all n. Since B_X is w-compact, we have $\bigcap_{n=1}^\infty \overline{C_n} \ne \emptyset$, say it contains x. Since $x \in \overline{C_1}$, there exists $y \in C_1$ such that $\|x - y\| < \frac{\theta}{3}$. Choose n such that $y \in \mathrm{conv}\{x_1, x_2, \dots x_n\}$. Since $x \in \overline{C_n}$, there exists $z \in C_n$ such that $\|x - z\| < \frac{\theta}{3}$. Then

$$\theta \le d(\mathrm{conv}\{x_1, \dots, x_n\}, \mathrm{conv}\{x_{n+1}, x_{n+2}, \dots\}) \le \|y - z\| \le \frac{2\theta}{3},$$

a contradiction. □

Ultrafilters

Fix a set $I \ne \emptyset$. A *filter* on I is a family $\mathcal{F} \subset \mathcal{P}(I)$ such that

(i) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$;

(ii) $A \subset B \subset I, A \in \mathcal{F} \implies B \in \mathcal{F}$;

(iii) $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$.

Remark. One can think of $\mathcal{F}$ as "big sets", or "full-measure".

Example. (i) For $i \in I$, $\mathcal{U}_i = \{A \subset I : i \in A\}$ is a filter – the *principal filter* at i.

(ii) If $|I| = \infty$, then $\{A \subset I : |I \setminus A| < \infty\}$ is a filter – the *cofinite filter* on I.

Definition. If X is a topological space, $f\colon I \to X$ is a function, $\mathcal{F}$ is a filter on I, $x \in X$, then we write $x = \lim_{\mathcal{F}} f$ if for all neighbourhoods U of x in X, $\{i \in I : f(i) \in U\} \in \mathcal{F}$.

Example.

(i) If $I = \mathbb{N}$, $\mathcal{F} =$ cofinite filter on $\mathbb{N}$, then this is just the usual notion of convergence of a sequence.

(ii) If X is Hausdorff and $x = \lim_{\mathcal{F}} f, y = \lim_{\mathcal{F}} f$, then $x = y$.

(iii) If $\mathcal{F} = \mathcal{U}_i$ for some $i \in I$, then $f(i) = \lim_{\mathcal{F}} f$ holds for all $f\colon I \to X$.

Definition. Let $I \neq \emptyset$ be a set. An *ultrafilter* on I is a maximal filter on I with respect to inclusion: it's a filter $\mathcal{U}$ such that if $\mathcal{F}$ is a filter and $\mathcal{U} \subset \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$.

Example. Any principal filter $\mathcal{U}_i = \{A \subset I : i \in A\}$ is an ultrafilter. If I is finite, then these are the only ones.

In general, any filter is contained in an ultrafilter (use Zorn's lemma).

Definition. A *free ultrafilter* is an ultrafilter that is not a principal ultrafilter.

Example. Any ultrafilter containing the cofinite filter is a free ultrafilter ($|I| = \infty$).

Lemma 6.4. Let $\mathcal{U}$ be an ultrafilter. If $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Proof. Assume otherwise, that $\exists C, D \in \mathcal{U}$ such that $A \cap C = B \cap D = \emptyset$. Then $(A \cup B) \cap (C \cap D) = \emptyset$, a contradiction, as $A \cup B, C \cap D \in \mathcal{U}$.

WLOG $A \cap C \neq \emptyset$ for all $C \in \mathcal{U}$. Then

$$\{D \subset I : \exists C \in \mathcal{U}, D \supset A \cap C\}$$

is a filter on I and it contains $\mathcal{U}$, so equals $\mathcal{U}$. So $A \in \mathcal{U}$. □

Remark.

(i) Every free ultrafilter contains the cofinite filter. [For any finite set $A \subset I$, consider $A \cup A^c$ in the lemma above.]

(ii) For an ultrafilter $\mathcal{U}$, define $\mu\colon \mathcal{P}(I) \to \{0, 1\}$ by $\mu(A) = 1_{A \in U}$. Then μ is a finitely additive measure.

Lemma 6.5. Let $\mathcal{U}$ be an ultrafilter and K be a compact topological space. Then for every function $f\colon I \to K$ there exists $x \in K$ such that $x = \lim_{\mathcal{U}} f$ (might not be unique, but if K is Hausdorff then it is). In particular, for every bounded function $f\colon I \to \mathbb{R}$ there exists a unique $x \in \mathbb{R}$ such that $x = \lim_{\mathcal{U}} f$.

Proof. If not, then $\forall x \in K$, $\exists$ open neighbourhood V_x of x such that $A_x = \{i \in I : f(i) \in V_x\} \notin \mathcal{U}$. Since K is compact, there exists a finite $F \subset K$ such that $\bigcup_{x \in F} V_x = K$. Then $\bigcup_{x \in F} A_x = I \in \mathcal{U}$ and by Lemma 4, $\exists x \in F$ such that $A_x \in \mathcal{U}$, a contradiction. □

Remark. Given bounded functions $f, g\colon I \to \mathbb{R}$ we have

$$\lim_{\mathcal{U}}(f + g) = \lim_{\mathcal{U}} f + \lim_{\mathcal{U}} g,$$
$$\lim_{\mathcal{U}}(fg) = \Big(\lim_{\mathcal{U}} f\Big)\Big(\lim_{\mathcal{U}} g\Big),$$

and if $f(i) \leq g(i)$ for all $i \in I$, then

$$\lim_{\mathcal{U}} f \leq \lim_{\mathcal{U}} g.$$

Ultraproduct and Ultrapowers

Definition. Fix a non-empty set I. We are given Banach spaces X_i, $i \in I$. We fix an ultrafilter $\mathcal{U}$ on I. We let

$$\left(\bigoplus_{i\in I} X_i\right)_\infty = \left\{(x_i)_{i\in I} : x_i \in X_i \,\forall i \in I,\ \sup_{i\in I}\|x_i\| < \infty\right\}.$$

This is a Banach space with norm $\|(x_i)\|_\infty = \sup_{i\in I}\|x_i\|$. Define

$$\|(x_i)\|_{\mathcal{U}} = \lim_{\mathcal{U}}\|x_i\|.$$

This defines a seminorm on $\left(\bigoplus_{i\in I} X_i\right)_\infty$. It follows that

$$\mathcal{N}_{\mathcal{U}} = \{(x_i) : \|(x_i)\|_{\mathcal{U}} = 0\}$$

is a subspace of $\left(\bigoplus_{i\in I} X_i\right)_\infty$, and the quotient $\left(\bigoplus_{i\in I} X_i\right)_\infty/\mathcal{N}_{\mathcal{U}}$ becomes a normed space with norm $\|((x_i)_{i\in I})_{\mathcal{U}}\| = \|(x_i)_{i\in I}\|_{\mathcal{U}}$ where for $x \in \left(\bigoplus_{i\in I} X_i\right)_\infty$, $x_{\mathcal{U}} = x + \mathcal{N}_{\mathcal{U}}$. It is easy to check that this is a complete norm. This Banach space is denoted by $\left(\prod_{i\in I} X_i\right)_{\mathcal{U}}$ — called an *ultraproduct* of $(X_i)_{i\in I}$.

If $X_i = X$ for all $i \in I$ for some Banach space X, then the ultraproduct $\left(\prod_{i\in I} X_i\right)_{\mathcal{U}}$ is denoted by $X^{\mathcal{U}}$ — called an *ultrapower* of X.

Proposition 6.6. Any ultrapower $X^{\mathcal{U}}$ of a Banach space X is finitely representable in X.

Proof. Let E be a finite-dimensional subspace of $X^{\mathcal{U}}$. Choose a basis $e_1, e_2, \dots, e_n$ of E. For each $1 \le k \le n$, fix $(x_{k,i})_{i\in I}$, a bounded sequence in X, such that $e_k = ((x_{k,i})_i)_{\mathcal{U}}$. So $\forall (\lambda_k)_{k=1}^n$ in $\mathbb{R}^n$, $\sum \lambda_k e_k = ((\sum \lambda_k x_{k,i})_i)_{\mathcal{U}}$.

Fix $\epsilon > 0$. We seek an injective linear map $T\colon E \to X$ such that $\|T\|\cdot\|T^{-1}\| < 1+\epsilon$ (here $T^{-1}\colon T(E) \to E$). Choose $\delta \in (0, \frac{1}{3})$ such that $\frac{1+\delta}{1-3\delta} < 1+\epsilon$. Let $S \subset \mathbb{R}^n$ be a finite set such that $\tilde{S} = \{\sum_{k=1}^n \lambda_k e_k : (\lambda_k)_{k=1}^n \in S\}$ is a δ-net of S_E.

Since $\|\sum_{k=1}^n \lambda_k e_k\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|\sum_{k=1}^n \lambda_k x_{k,i}\| = 1$ for all $(\lambda_k) \in S$, we have

$$\left\{ i \in I : 1-\delta < \left\|\sum_{k=1}^n \lambda_k x_{k,i}\right\| < 1+\delta \right\} \in \mathcal{U}.$$

Since S is finite, these sets have intersection in $\mathcal{U}$. In particular, $\exists i_0 \in I$ such that

$$1-\delta < \left\|\sum_{k=1}^n \lambda_k x_{k,i_0}\right\| < 1+\delta \qquad \forall (\lambda_k) \in S.$$

Now define $T\colon E \to X$, $T(\sum_{k=1}^n \mu_k e_k) = \sum_{k=1}^n \mu_k x_{k,i_0}$, $(\mu_k) \in \mathbb{R}^n$. Given $x \in S_E$, $\exists z \in \tilde{S}$ such that $\|x - z\| \le \delta$. So

$$\|Tx\| \le \|Tz\| + \|T(x-z)\| \le (1+\delta) + \|T\|\delta.$$

Taking sup over $x \in S_E$, $\|T\| \le 1+\delta+\delta\|T\|$, so $\|T\| \le \frac{1+\delta}{1-\delta}$. It follows that $\|Tx\| \ge \|Tz\| - \|T(x-z)\| \ge 1-\delta-\frac{1+\delta}{1-\delta}\delta = \frac{1-3\delta}{1-\delta}$. Hence $\left\|T^{-1}\right\| \le \frac{1-\delta}{1-3\delta}$ and $\|T\|\,\|T^{-1}\| \le \frac{1+\delta}{1-3\delta} < 1+\epsilon$. $\square$

Theorem 6.7. Let X be a Banach space. Then X is superreflexive $\iff$ whenever Y is crudely finitely representable in X, then Y is reflexive.

Proof. ($\Longleftarrow$): clear from definition. ($\Longrightarrow$): assume Y is non-reflexive and crudely finitely representable in X. Fix $\theta \in (0,1)$. By Theorem 3, $\exists (y_i)_{i=1}^\infty$ in B_Y such that $\forall n$,

$$d(\mathrm{conv}(y_1, \dots, y_n), \mathrm{conv}(y_{n+1}, y_{n+2}, \dots)) \geq \theta.$$

There exists $\lambda > 1$ such that $\forall$ subspace $E \subset Y$, $\dim E < \infty$, $\exists$ linear $T \colon E \to X$ such that

$$\lambda^{-1} \|y\| \leq \|Ty\| \leq \|y\| \qquad \forall y \in E.$$

For $N \in \mathbb{N}$, $\exists$ linear map $T_N \colon \mathrm{span}(y_1, \dots, y_N) \to X$ such that

$$\lambda^{-1} \|y\| \leq \|T_N y\| \leq \|y\| \qquad \forall y \in \mathrm{span}(y_1, \dots, y_N).$$

Let $x_{N,i} = T_N(y_i)$ for $1 \leq i \leq N$. Note that for $1 \leq m < n \leq N$ and for convex combinations $\sum_{i=1}^m t_i x_{N,i}$, $\sum_{i=m+1}^n t_i x_{N,i}$, we have

$$\left\| \sum_{i=1}^m t_i x_{N,i} - \sum_{i=m+1}^n t_i x_{N,i} \right\| \geq \frac{1}{\lambda} \left\| \sum_{i=1}^m t_i y_i - \sum_{i=m+1}^n t_i y_i \right\| \geq \frac{\theta}{\lambda}.$$

Note also that $\|x_{N,i}\| \leq 1$ for all $1 \leq i \leq N$. WLOG replace θ/λ by θ. Now fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Define

$$\tilde{x}_{N,i} = \begin{cases} x_{N,i} & \text{if } i \leq N \\ 0 & \text{if } i > N, \end{cases} \qquad \tilde{x}_i = ((\tilde{x}_{N,i})_{N=1}^\infty)_{\mathcal{U}}.$$

Given $1 \leq m < n$ and convex combinations $z = \sum_{i=1}^m t_i \tilde{x}_i$ and $w = \sum_{i=m+1}^n t_i \tilde{x}_i$ in $X^{\mathcal{U}}$, we have $\forall N \in \mathbb{N}, N \geq n$,

$$\left\| \sum_{i=1}^m t_i \tilde{x}_{N,i} - \sum_{i=m+1} t_i \tilde{x}_{N,i} \right\| \geq \theta.$$

It follows that $\|z - w\| \geq \theta$. Then

$$d(\mathrm{conv}\{\tilde{x}_1, \dots, \tilde{x}_m\}, \mathrm{conv}\{\tilde{x}_{m+1}, \dots\}) \geq \theta.$$

By Theorem 3, $X^{\mathcal{U}}$ is non-reflexive. By Proposition 6, $X^{\mathcal{U}}$ is finitely representable in X, and hence X is not superreflexive. $\square$

Definition. A Banach space X is *strictly convex* if $\forall x, y \in S_X$, $x \neq y$, $\left\| \frac{x+y}{2} \right\| < 1$. Say X is *uniformly convex* if $\forall \epsilon \in (0,2]$, $\exists \delta > 0$, $\forall x, y \in S_X$, $\|x - y\| \geq \epsilon \implies 1 - \left\| \frac{x+y}{2} \right\| \geq \delta$. The *modulus of uniform convexity* of X is the function $\delta_X \colon [0,2] \to \mathbb{R}^+$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \|x - y\| \geq \epsilon\}.$$

Example. (i) ℓ_2 is uniformly convex: given $x, y \in S_{\ell_2}$ with $\|x - y\| \geq \epsilon$, we have, by the parallelogram rule,

$$4 = 2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \geq \|x+y\|^2 + \epsilon^2.$$

So $1 - \left\| \frac{x+y}{2} \right\| \geq 1 - \sqrt{1 - \frac{\epsilon^2}{4}} \approx \frac{\epsilon^2}{8}$.

(ii) Choose $1 < p_n < 2$, $p_n \to 1$. Let $X = \left(\bigoplus_{n=1}^{\infty} \ell_{p_n}^2\right)_{\ell_2}$. Then X is strictly convex, but not uniformly convex. However, $X \sim (\bigoplus_{n=1}^{\infty} \ell_2^n)_{\ell_2} \cong \ell_2$. So uniform convexity is not an isomorphic property.

(iii) c_0, ℓ_1, ℓ_∞ are not strictly convex.

Theorem 6.8 (Milman-Pettis)**.** If X is uniformly convex, then X is reflexive.

Remark. Recall Goldstine's Theorem: $\overline{B_X}^{w^*} = B_{X^{**}}$. In fact, if $\dim X = \infty$, then $\overline{S_X}^{w^*} = B_{X^{**}}$.

Proof. Let $\varphi \in B_{X^{**}}$ and U be a w^*-neighbourhood of φ. WLOG $\exists n \in \mathbb{N}$, $f_1, \dots, f_n \in X^*$, $\epsilon > 0$ such that $U = \{\psi \in X^{**} : |(\psi - \varphi)(f_i)| < \epsilon, \forall i\}$. Choose $x \in B_X \in U$ by Goldstine. Fix $z \in \bigcap_{i=1}^n \ker f_i$, $z \neq 0$ ($\dim X = \infty$). Then $x + \lambda z \in U$ $\forall \lambda \in \mathbb{R}$, and $\exists \lambda \in \mathbb{R}$ such that $\|x + \lambda z\| = 1$. □

Proof of Theorem 8. WLOG $\dim X = \infty$. Fix $\varphi \in S_{X^{**}}$. We show that $\varphi \in X$. Then we'll be done. Fix $\epsilon \in (0, 2)$ and let $\delta = \delta_X(\epsilon) > 0$. Then $\forall x, y \in S_X$ if $\|x + y\| \geq 2 - \delta$, then $1 - \left\|\frac{x+y}{2}\right\| \leq \frac{\delta}{2} < \delta$, and hence $\|x - y\| < \epsilon$. Choose $f_\epsilon \in B_{X^*}$ such that $\varphi(f_\epsilon) > 1 - \frac{\delta}{2}$. Let $V_\epsilon = \{\psi \in X^{**} : \psi(f_\epsilon) \geq 1 - \frac{\delta}{2}\}$. This is a w^*-closed neighbourhood of φ. Hence $W_\epsilon = V_\epsilon \cap S_X$ is non-empty and $\|\cdot\|$-closed subset of X. Also, given $x, y \in W_\epsilon$, $\|x + y\| \geq f_\epsilon(x + y) \geq 2 - \delta$, and hence $\|x - y\| < \epsilon$. Thus, $\operatorname{diam}(W_\epsilon) \leq \epsilon$. Now for $n \in \mathbb{N}$, let

$$A_n = \bigcap_{k=1}^{n} W_{1/k} = \{\psi \in X^{**} : \psi(f_{1/k}) \geq 1 - \frac{\delta_X(1/k)}{2} \text{ for } k = 1, \dots, n\} \cap S_X.$$

So A_n is a non-empty, $\|\cdot\|$-closed subset of X of diameter at most $\operatorname{diam}(W_{1/n}) \leq \frac{1}{n}$. Also, $A_n \supset A_{n+1}$ for all n, and X is complete, so by Cantor's intersection Theorem, $\cap_{n=1}^{\infty} A_n = \{x\}$ for some $x \in S_X$.

We show that $\varphi = \hat{x}$. If not, then $\exists g \in X^*$, $\eta = \varphi(g) - g(x) > 0$. Let

$$\begin{aligned} B_n &= A_n \cap \{\psi : |\varphi(g) - \psi(g)| \leq \frac{\eta}{2}\} \\ &= \underbrace{\{\psi : \psi(f_{1/k}) \geq 1 - \frac{\delta_X(1/k)}{2} \text{ for } k = 1, \dots, n, \ |\varphi(g) - \psi(g) \leq \frac{\eta}{2}\}}_{w^*\text{-closed neighbourhood of } \varphi} \cap S_X, \end{aligned}$$

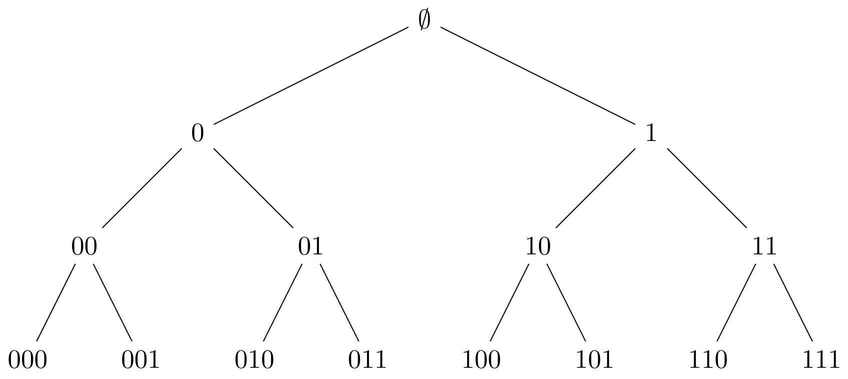
so B_n is nonempty, $\|\cdot\|$-closed and $\operatorname{diam}(B_n) \leq \operatorname{diam}(A_n) \to 0$. So $\cap_{n=1}^{\infty} B_n = \{x\}$, so $|\varphi(g) - g(x)| \leq \frac{\eta}{2}$, a contradiction. □

Fact (Enflo)**.** $(X, \|\cdot\|)$ is superreflexive $\iff$ $\exists$ equivalent norm $\|\cdot\|'$ on X such that $(X, \|\cdot\|')$ is uniformly convex. Recall norm equivalence means $\exists a, b > 0$ such that

$$a\|x\| \leq \|x\|' \leq b\|x\|.$$

Example. $\ell_2 \oplus_2 \ell_1^2 \sim \ell_2 \oplus_2 \ell_2^2 \cong \ell_2$, which is superreflexive but $\ell_2 \oplus_2 \ell_1^2$ is not strictly convex.

Recall that the *binary tree* of depth n, B_n, has vertex set $\cup_{k=0}^{n} \{0, 1\}^k$ and $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{0, 1\}^k$, $k < n$, is joined to $(\epsilon_1, \dots, \epsilon_k, i)$, $i = 0, 1$.



Notation. Given $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$, $\delta = (\delta_1, \ldots, \delta_\ell)$, we write $\epsilon \preceq \delta$ if $k \leq \ell$ and $\epsilon_i = \delta_i$ for $1 \leq i \leq k$. We also let $|\epsilon| = k$ denote the *length of* ϵ.

Definition. We say a Banach space X has the *finite tree property* if $\exists \theta > 0$, $\forall n \in \mathbb{N}$, $\exists \{x_\epsilon : \epsilon \in B_n\} \subset B_X$ such that $x_\epsilon = \frac{1}{2}(x_{\epsilon 0} + x_{\epsilon 1})$ for all $\epsilon \in B_{n-1}$, $\|x_\epsilon - x_{\epsilon i}\| \geq \theta$ $\forall \epsilon \in B_{n-1}$, $i = 0, 1$.

Theorem 6.9. For a Banach space, the following are equivalent:

(a) X is not superreflexive;

(b) X has the finite tree property;

(c) $\exists \theta > 0$, $\forall n \in \mathbb{N}$, $\exists \{x_1, \ldots, x_n\} \subset B_X$ such that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geq \theta \left| \sum_{i=\ell}^{m} a_i \right| \quad \forall a_1, \ldots, a_n \in \mathbb{R}, \quad 1 \leq \ell \leq m \leq n.$$

Remark. Let $S = \{(a_i)_{i=1}^\infty \subset \mathbb{R} : \sum_{i=1}^\infty a_i \text{ is convergent}\}$. This becomes a normed space with

$$\|(a_i)\| = \sup \left\{ \left| \sum_{i=\ell}^{m} a_i \right| : 1 \leq \ell \leq m \right\}.$$

This is called the *summing norm*. Note $S \sim c_0$, via the map

$$(a_i)_{i=1}^\infty \longmapsto \left(\sum_{i=n}^{\infty} a_i \right)_{n=1}^{\infty}.$$

Definition. Given a convex set C in a Banach space Z, a point $w \in C$ is *strongly exposed* if $\exists f \in Z^*$ such that

(i) $f(u) < f(w) \quad \forall u \in C$, $u \neq w$;

(ii) $\operatorname{diam}\{u \in C : f(w) - \epsilon < f(u)\} \to 0$ as $\epsilon \to 0$.

Theorem 6.10. Every non-empty, w-compact convex subset of a separable Banach space has a strongly exposed point.

Proof. Omitted. Theorem is also true for non-separable spaces. □

Proof of Theorem 9. (a) $\Longrightarrow$ (b): There exists a non-reflexive Z finitely representable in X. Fix $\theta \in (0,1)$. By Theorem 3, $\exists (z_i)$ in B_Z such that $d(\operatorname{conv}\{z_1,\dots,z_n\}, \operatorname{conv}\{z_{n+1},\dots\}) \geq \theta$ for all $n \in \mathbb{N}$. For $\epsilon = (\epsilon_1,\dots,\epsilon_n) \in B_n$, let $k(\epsilon) = 1 + \sum_{i=1}^n 2^{n-i}\epsilon_i$. This is an enumeration of the leaves. For $\delta \in B_n$, let $I_\delta = \{k(\epsilon) : \delta \preceq \epsilon, |\epsilon| = n\}$, set of nth generation descendants of δ. Let $z_\delta = 2^{|\delta|-n}\sum_{k\in I_\delta} z_k$. Since $|I_\delta| = 2^{n-|\delta|}$, we have $z_\delta \in \operatorname{conv}\{z_k : k \in I_\delta\} \subset B_Z$. For $\delta \in B_{n-1}$, $I_\delta = I_{\delta,0} \cup I_{\delta,1}$ and $I_{\delta,0} \cap I_{\delta,1} = \emptyset$, and moreover, $\forall k \in I_{\delta,0}$, $\forall \ell \in I_{\delta,1}$, $k < \ell$. It follows that $z_\delta = \frac{1}{2}(z_{\delta 0} + z_{\delta 1})$, and for $i = 0,1$, we have $\|z_\delta - z_{\delta,i}\| = \frac{1}{2}\|z_{\delta 0} - z_{\delta 1}\| \geq \frac{1}{2} d(\operatorname{conv}\{z_k : k \in I_{\delta 0}\}, \operatorname{conv}\{z_k : k \in I_{\delta 1}\}) \geq \frac{\theta}{2}$. So Z has the finite tree property, and hence so does X since Z is finitely representable in X.

(b) $\Longrightarrow$ (a): $\exists \theta > 0$, $\forall n$, $\exists \{x_\epsilon^n : \epsilon \in B_n\} \subset B_X$ such that $x_\epsilon^n = \frac{1}{2}(x_{\epsilon 0}^n + x_{\epsilon 1}^n)$ $\forall \epsilon \in B_{n-1}$ and $\|x_\epsilon^n - x_{\epsilon i}^n\| \geq \theta$ $\forall \epsilon \in B_{n-1}$, $i = 0,1$. Let $\mathcal{U}$ be a free ultrafilter and let B_∞ be the ∞ binary tree with vertex set $\bigcup_{k=0}^\infty \{0,1\}^k$ and ϵ joined to ϵi $\forall \epsilon \in B_\infty$, $i = 0,1$. Let

$$\tilde{x}_\epsilon^n = \begin{cases} x_\epsilon^n & \text{if } |\epsilon| \leq n \\ 0 & \text{if } n < |\epsilon|. \end{cases} \quad \text{and} \quad \tilde{x}_\epsilon = ((\tilde{x}_\epsilon^n)_n)_{\mathcal{U}}.$$

It's easy to see that $\tilde{x}_\epsilon = \frac{1}{2}(\tilde{x}_{\epsilon 0} + \tilde{x}_{\epsilon 1})$ and $\|\tilde{x}_\epsilon - \tilde{x}_{\epsilon i}\| \geq \theta$ $\forall \epsilon \in B_\infty$, $i = 0,1$. Let $Z = \overline{\operatorname{span}}\{\tilde{x}_\epsilon : \epsilon \in B_\infty\}$. This is a separable subspace of $X^{\mathcal{U}}$. Assume for contradiction that X is superreflexive. Then by Proposition 6, Z is reflexive. Then B_Z is w-compact. Let $C = \overline{\operatorname{conv}}\{\tilde{x}_\epsilon : \epsilon \in B_\infty\}$. Then C is a $\|\cdot\|$-closed convex subset of B_Z, and hence w-compact. By Theorem 10, C has a strongly exposed point w. So $\exists f \in Z^*$ such that $f(u) < f(w)$ $\forall u \in C, u \neq w$ and $\exists \eta > 0$ $\{u \in C : f(u) > f(w) - \eta\}$ has diameter $< \frac{\theta}{2}$. Since $\{u \in C : f(u) \leq f(w) - \eta\}$ is $\|\cdot\|$-closed and convex and $\subsetneq C$, it cannot contain $\tilde{x}_\epsilon$ $\forall \epsilon$. So $\exists \epsilon \in B_\infty$ such that $f(\tilde{x}_\epsilon) > f(w) - \eta$. Then $\frac{1}{2}(f(\tilde{x}_{\epsilon 0}) + f(\tilde{x}_{\epsilon 1})) = f(\tilde{x}_\epsilon)$, so $\exists i \in \{0,1\}$ such that $f(\tilde{x}_{\epsilon i}) > f(w) - \eta$. Thus $\|\tilde{x}_\epsilon - \tilde{x}_{\epsilon i}\| < \frac{\theta}{2}$, a contradiction.

(a) $\Longrightarrow$ (c): Let Z be non-reflexive and finitely representable in X. By Theorem 2, $\exists \theta \in (0,1)$ and (z_i) in B_Z, (h_i) in B_{Z^*} such that

$$h_i(z_j) = \begin{cases} \theta & i \leq j \\ 0 & i > j. \end{cases}$$

Given scalars $(a_i)_{i=1}^n$, $|\sum_{i=\ell}^n a_i| = \left|\frac{1}{\theta} h_\ell\left(\sum_{i=1}^n a_i z_i\right)\right| \leq \frac{1}{\theta}\|\sum_{i=1}^n a_i z_i\|$. If $1 \leq \ell \leq m \leq n$, then

$$\left|\sum_{i=\ell}^m a_i\right| \leq \left|\sum_{i=\ell}^n a_i\right| + \left|\sum_{i=m+1}^n a_i\right| \leq \frac{2}{\theta}\left\|\sum_{i=1}^n a_i z_i\right\|.$$

Since Z is finitely representable in X, $\forall \lambda > \frac{2}{\theta}$, $\forall n$, $\exists x_1,\dots,x_n \in B_X$ such that

$$\left|\sum_{i=\ell}^m a_i\right| \leq \lambda\left\|\sum_{i=1}^n a_i x_i\right\| \qquad \forall a_1,\dots,a_n \in \mathbb{R}, 1 \leq \ell \leq m.$$

(c) $\Longrightarrow$ (a):$\exists \theta > 0$, $\forall n \in \mathbb{N}$, $\exists \{x_1^n,\dots,x_n^n\} \subset B_X$ such that

$$\left\|\sum_{i=1}^n a_i x_i^n\right\| \geq \theta\left|\sum_{i=\ell}^m a_i\right| \qquad \forall a_1,\dots,a_n \in \mathbb{R}, \quad 1 \leq \ell \leq m \leq n.$$

Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the usual process yields an infinite sequence $(\tilde{x}_i)_{i=1}^\infty$ in $B_{X^\mathcal{U}}$ such that $\forall n \in \mathbb{N}$, $\forall a_1, \dots, a_n \in \mathbb{R}$, $\forall 1 \le \ell \le m \le n$,

$$\left\| \sum_{i=1}^n a_i \tilde{x}_i \right\| \ge \theta \left| \sum_{i=\ell}^m a_i \right|.$$

It follows that $\forall i \in \mathbb{N}$,

$$h_i(\tilde{x}_j) = \begin{cases} \theta & i \le j \\ 0 & i > j \end{cases}$$

extends to a well-defined linear functional on $X^\mathcal{U}$ with $\|h_i\| \le 1$ [also uses Hahn-Banach]. By Theorem 3, $X^\mathcal{U}$ is not reflexive. By Proposition 6, $X^\mathcal{U}$ is finitely representable in X, so X is not superreflexive. □

Theorem 6.11 (Metric Characterization of Superreflexivity)**.** Let X be a Banach space. The following are equivalent:

(a) X not superreflexive;

(b) The sequence (D_n) of diamond graphs embeds uniformly bilipschitzly into X.

Sketch proof. (non-examinable) (b) $\implies$ (a): Have $f_n \colon D_n \to X$ $\sup_n \operatorname{dist}(f_n) < \infty$. WLOG $\exists \delta > 0$, $\forall n$, $\forall x, y \in D_n$, $\delta 2^{-n} d_n(x, y) \le \|f_n(x) - f_n(y)\| \le 2^{-n} d_n(x, y)$. Let $D_0 = tb$, $D_1 = tb\ell r$, and D_n is a union of 4 copies of D_{n-1}. Fix n, $f = f_n$. Let $x_\emptyset = f(t) - f(b)$. Then $\|x_\emptyset\| \le 2^{-n} d_n(t, b) = 1$. Consider $\|[(f(t) - f(\ell)) - (f(\ell) - f(b))] - [(f(t) - f(r)) - (f(r) - f(b))]\| = \|2(f(r) - f(\ell))\| \ge 2\delta 2^{-n} d_n(\ell, r) = 2\delta$. WLOG $\|(f(t) - f(\ell)) - (f(\ell) - f(b))\| \ge \delta$. Let $x_0 = 2(f(\ell) - f(b))$, $x_1 = 2(f(t) - f(\ell))$. Then $x_\emptyset = \frac{1}{2}(x_0 + x_1)$ and $\|x_\emptyset - x_0\| = \frac{1}{2}\|x_1 - x_0\| \ge \delta$. Continue inductively.

(a) $\implies$ (b): $\exists \theta > 0$, $\forall n$, $\exists x_1, \dots, x_{2^n} \in B_X$ with lower summing norm estimate. First embed $f_n \colon D_n \to \{0, 1\}^{2^n} \subset \ell_1^{2^n}$. For D_0, do $t = 1, b = 0$. For D_1, do $t = 11, \ell = 01, b = 00, r = 10$. If $xy \in E_{n-1}$, $f_{n-1}(x), f_{n-1}(y) \in \{0, 1\}^{2^{n-1}}$ differ in one digit, say j. Consider $yuxv$ in D_n. If $\nu \in \{x, y, u, v\}$, $(f_n(\nu))_{2i-1} = (f_n(\nu))_{2i} = (f_{n-1}(x))_i$. $f_n(\nu)_{2j-1}, f_n(\nu)_{2j}$ will be $00, 11, 01, 10$ for $\nu = x, y, u, v$ $(f_{n-1}(x))_j = 0$.

Let $g_n \colon D_n \to X$ given by

$$g_n(x) = \sum_{j=1}^{2^n} \epsilon_j x_j, \qquad (\epsilon_j) = f_n(x).$$

If x is in top left, y is in bottom right, then$f_n(x) = (f_{n-1}(x), \underbrace{1, \dots, 1}_{2^{n-1}})$, $f_n(y) = (f_{n-1}(y), \underbrace{0, \dots, 0}_{2^{n-1}})$. □

Exam will be 4 questions, answer 3 in 3 hours. Mostly bookwork.