NSTIA Mathematics, course B **Stuart B. Dalziel**
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Mathematical Methods I: Examples I

This is the first (and longer) of two examples sheets for the Michaelmas term course. You are not necessarily expected to attempt all the questions, supervisors will be able to advise about selection of questions.

In addition to the *main questions*, some sections contain *basic skills* questions that are either revision of material from your A-level courses or an opportunity to practise new concepts with rather straightforward manipulations only. Numerical answers to the basic-skills questions are included at the end of this sheet; those for the main questions will be circulated after the middle of term. We may discuss some questions in lectures.

Comments and suggestions are welcome (please email to *smc1@cam.ac.uk*).

A. Vector addition and subtraction

Basic skills

A1. The displacement vectors $\vec{OA} = (2,3)$ and $\vec{AB} = (4,6)$. What is the displacement vector $\vec{BO}$?

A2. Consider three points A, B and C with coordinates $(2,3,1)$, $(6,2,5)$ and $(3,3,8)$ respectively. Which pair of points are closest together and what is their relative displacement?

Main questions

A3. An aeroplane has an air velocity of 125 $\mathrm{km\,h^{-1}}$ due North. How fast will it travel over the earth and in what direction if the wind is: (i) from the North at speed 40 $\mathrm{km\,h^{-1}}$; (ii) from the West at speed 50 $\mathrm{km\,h^{-1}}$; and (iii) from the South-East at speed 80 $\mathrm{km\,h^{-1}}$?

A4. Prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

A5. ABC is a triangle whose vertices have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

i. What is the position vector $\mathbf{d}$ of the mid-point of BC?

ii. Write down the position vector $\mathbf{p}$ of a point P on the *median* joining A to the mid-point of BC and a fraction λ of the way along it.

iii. Write down similar expressions for points on the other two medians.

iv. By guessing a suitable value of λ (or otherwise) show that the three medians all meet in one point. What is its position vector?

A6. Show that for any tetrahedron, the lines joining the mid-points of opposite edges are concurrent.

A7. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar vectors related by $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{0}$, where λ, μ and ν are non-zero, show that the condition for the points with position vectors $\alpha\mathbf{a}$, $\beta\mathbf{b}$ and $\gamma\mathbf{c}$ to be collinear is

$$\frac{\lambda}{\alpha} + \frac{\mu}{\beta} + \frac{\nu}{\gamma} = 0\,.$$

A8. Show that the points with position vectors $(1, 0, 1)$, $(1, 1, 0)$ and $(1, -3, 4)$ lie on a straight line and find the equation of this line in the two forms:

i. $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$; and

ii. $\frac{x-x_0}{c} = \frac{y-y_0}{d} = \frac{z-z_0}{e}$.

B. Scalar product

Basic skills

B1. The vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$. Calculate the scalar products $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{a}$ directly and verify that they are equal.

B2. The vectors $\mathbf{c} = (2, 1, 4)$ and $\mathbf{d} = (3, -2, 1)$.

i. Calculate the angle between the vectors $\mathbf{c}$ and $\mathbf{d}$.

ii. Find a vector perpendicular to $\mathbf{c}$ and calculate the angle it makes with $\mathbf{d}$.

iii. Find a vector perpendicular to $\mathbf{d}$ and calculate the angle it makes with $\mathbf{c}$.

Main questions

B3. The points A and B have position vectors $\mathbf{a} = (0, 3, 4)$ and $\mathbf{b} = (3, 2, 1)$ relative to the origin O. Find:

i. The lengths of OA, OB and AB;

ii. The angle AOB; and

iii. The angle between OA and AB.

B4. Four points, A, B, C and D, are such that $AD \perp BC$ and $BD \perp AC$. Show that $CD \perp AB$.

B5. What are the properties of the plane $\mathbf{b} \cdot (\mathbf{r} - \mathbf{a}) = 0$? If O is the origin and a point C has position vector $\mathbf{c}$, find:

i. the length of the projection of OC onto the plane; and

ii. the distance of C from the plane.

B6. From the inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ deduce the triangle inequality $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}|$.

B7. Identify the surfaces

i. $|\mathbf{r}| = k$;

ii. $\mathbf{r} \cdot \mathbf{u} = l$;

iii. $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$; and

iv. $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = n$,

where k, l, m and n are positive fixed scalars (with $m < 1$) and $\mathbf{u}$ is a fixed unit vector.

B8. Find scalars a and b such that the vectors $\mathbf{e}_1 = (1, 1, 0)$, $\mathbf{e}_2 = (1, a, 1)$ and $\mathbf{e}_3 = (1, b, -2)$ form an orthogonal basis. With this choice of a and b, find c, d and e such that

$$(c + d - 2e)\mathbf{e}_1 + (e - 1)\mathbf{e}_2 + (c + e)\mathbf{e}_3 = \mathbf{0} \,.$$

B9. Show that the line of intersection of the two planes $x+2y+3z = 0$ and $3x+2y+z = 0$ is equally inclined to the x- and z-axes and makes an angle $\cos^{-1}(-\sqrt{2/3})$ with the y-axis.

B10. Calculate the shortest distances between the plane $5x + 2y - 7z + 9 = 0$ and the points $(1, -1, 3)$ and $(3, 2, 3)$. Are the points on the same side of the plane?

B11. Prove that for any tetrahedron, the sum of the squares of the lengths of the edges equals four times the sum of the squares of the lengths of the lines joining the mid-points of opposite edges.

B12. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of equal length l, and define the positions of the points A, B and C relative to O. If O, A, B, and C are vertices of a regular tetrahedron, find the distance of any one vertex from the opposite face.

B13. Find the acute angle at which two diagonals of a cube intersect.

C. Vector product

Basic skills

C1. The vectors $\mathbf{a} = (2, 1, 3)$, $\mathbf{b} = (6, 0, 5)$ and $\mathbf{c} = (5, 3, 1)$ respectively. Evaluate the following:

i. $\mathbf{a} \times \mathbf{b}$ and show that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$;

ii. $\mathbf{b} \times \mathbf{a}$ and show that it is equal to $-(\mathbf{a} \times \mathbf{b})$;

iii. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ and show that it is equal to $\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$;

iv. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and show that it is *not* equal to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$;

v. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and show that it is equal to $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$; and

vi. $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ and show that it is equal to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Main questions

C2. Find the angle between the position vectors of the points $(2, 1, 1)$ and $(3, -1, -5)$, and find the direction cosines of a vector perpendicular to both.

C3. Show that for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

i. $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$; and

ii. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

C4. Describe the locus of the points that satisfy the equation $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (1, -1, 0)$.

C5. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar. The vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ defined by

$$\mathbf{A} \equiv \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \quad \mathbf{B} \equiv \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \quad \text{and} \quad \mathbf{C} \equiv \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]},$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, are often called "reciprocal vectors". Show that:

i. $\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$;

ii. $\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = 0$, etc.;

iii. $[\mathbf{A}, \mathbf{B}, \mathbf{C}] = 1/[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; and

iv. $\mathbf{a} = \mathbf{B} \times \mathbf{C}/[\mathbf{A}, \mathbf{B}, \mathbf{C}]$.

C6. In the notation of the previous question, show that the plane through the points $\mathbf{a}/\alpha$, $\mathbf{b}/\beta$ and $\mathbf{c}/\gamma$ is normal to the direction of the vector $\mathbf{g} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$. Hence show that the perpendicular distance of the plane from the origin in $1/|\mathbf{g}|$.

C7. You need to drill a hole in a piece of metal starting at a right angle to a flat surface passing through the points $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$, with the hole emerging at the point $D = (2, 1, 0)$. How long a drill must you use and where (in the plane ABC) must you start drilling?

C8. Find the distance between any vertex of a unit cube and a diagonal of the cube which does not pass through the vertex by:

i. using a cross product; and

ii. finding the perpendicular from the vertex to the diagonal.

C9. Find an equation of the form $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ for the plane which passes through the points $(1, 1, 1)$, $(1, 2, 3)$ and $(0, 0, 4)$.

C10. Show that the vectors $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (0, 0, 1)$ and $\mathbf{c} = (2, -1, 1)$ form a non-orthogonal basis and, using the scalar triple product, write the vector $\mathbf{d} = (1, 1, 1)$ in terms of this basis.

C11. Simplify $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

C12. Solve the vector equation $\mathbf{a} \times \mathbf{r} + \lambda\mathbf{r} = \mathbf{c}$ for $\mathbf{r}$, where $\lambda \neq 0$.

D. Vector area

D1. Given an origin $O = (0, 0, 0)$ and points $B = (2, 0, 0)$, $C = (0, 2, 0)$, $D = (2, 2, 0)$ and $E = (1, 1, 1)$, what are the vector areas of:

i. the projection of the square $OBDC$ onto the plane with unit normal $\hat{\mathbf{n}} = (0, -1, 1)/\sqrt{2}$;

ii. the upper surface area of the pyramid with base $OBDC$ and vertex E; and

iii. a lampshade (truncated hollow cone) bounded by a horizontal circle of radius 4 units and a horizontal circle of radius 3 at height 5 above the first?

D2. If $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$, find $\mathbf{s}$, the vector surface area of the loop $OABCO$ by:

i. drawing the loop projections onto the yz, zx and xy planes and thus finding the components of $\mathbf{s}$; and

ii. filling in the loop with two or three plane polygons, finding the vector area of each, and combining the results.

What is the projected area of the loop as seen from:

i. the direction $(0, -1, 1)$; and

ii. the direction which makes the projected area as large as possible?

E. Coordinate systems

Basic skills

E1. By drawing a 2-d diagram, find the Cartesian coordinates (x, y) of a point with plane-polar coordinates (r, ϕ).

E2. In 2-d Cartesian coordinates, a circle is specified by $(x-1)^2 + y^2 = 1$. Find its equation in plane-polar coordinates.

Main questions

E3. A point has Cartesian coordinates $(3, 4, 5)$. What are its spherical-polar and cylindrical-polar coordinates?

E4. Identify the following loci, where a is a constant:

i. plane-polar coordinates

$$\text{(a)}\quad \phi = a \qquad\qquad \text{(b)}\quad r = \phi\,;$$

ii. cylindrical-polar coordinates

$$\text{(a)}\quad z = a \qquad \text{(b)}\quad r = a \qquad \text{(c)}\quad r = a \quad \text{and} \quad z = \phi\,;$$

iii. spherical-polar coordinates

$$\text{(a)}\quad \theta = a \qquad \text{(b)}\quad \phi = a \qquad \text{(c)}\quad r = a$$
$$\text{(d)}\quad r = \theta = a \qquad \text{(e)}\quad r = \theta = \phi\,.$$

F. Complex numbers

Basic skills

F1. Find the real and imaginary parts of the following complex numbers (where n is an integer and ω, t, R, L and C are real):

$$\text{(a)}\quad i^3 \qquad \text{(b)}\quad i^{4n} \qquad \text{(c)}\quad \left(\frac{1+i}{\sqrt{2}}\right)^2 \qquad \text{(d)}\quad \left(\frac{1-i}{\sqrt{2}}\right)^2$$
$$\text{(e)}\quad \left(\frac{1+\sqrt{3}i}{2}\right)^3 \qquad \text{(f)}\quad \frac{1+i}{2-5i} \qquad \text{(g)}\quad \left(\frac{1+i}{1-i}\right)^2 \qquad \text{(h)}\quad \frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}}\,.$$

F2. Plot the following complex numbers on an Argand diagram:

$$z_1 = 2 + i\,, \quad z_2 = 3 + 4i\,, \quad z_1^*\,, \quad z_2^*\,, \quad z_2 - z_1\,, \quad z_2 - 2z_1\,.$$

Find $z_1 z_2$ by: (i) adding arguments and multiplying moduli; and (ii) by using the rules of complex algebra.

F3. Find the modulus and principal argument of

$$\text{(a)} \quad 1 + \sqrt{3}i \qquad \text{(b)} \quad -1 + i \qquad \text{(c)} \quad -\sqrt{3} - \frac{i}{\sqrt{3}} \qquad \text{(d)} \quad \frac{e^{i\omega t}}{R + i\omega L + (i\omega C)^{-1}}\,,$$

where ω, t, R, C and L are real.

F4. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:

$$\text{(a)} \quad z^2 \qquad \text{(b)} \quad iz \qquad \text{(c)} \quad (1+i)z \qquad \text{(d)} \quad z^2(z-1)\,.$$

Main questions

F5. Factorise the following expressions:

$$\text{(a)} \quad z^2 + 1 \qquad \text{(b)} \quad z^2 - 2z + 2 \qquad \text{(c)} \quad z^2 + i \qquad \text{(d)} \quad z^2 + (1-i)z - i\,.$$

F6. By taking the complex conjugate of the equation, show that the solutions to

$$z^2 + az + b = 0 \qquad (a \text{ and } b \text{ real})$$

are either real or come in complex-conjugate pairs. What can be said about nth-order polynomial equations with real coefficients [i.e. equations of the form

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0 \qquad (a_0,\, a_1,\, \ldots,\, a_n \text{ real})]?$$

F7. Describe the set of points in the complex plane described by the following equations:

(a) $\lvert z\rvert = 4$	(b) $\lvert z-1\rvert = 3$	(c) $\lvert z-i\rvert = 2$
(d) $\lvert z-(1-2i)\rvert = 3$	(e) $\lvert z^*-1\rvert = 1$	(f) $\lvert z^*-i\rvert = 1$
(g) $\lvert z-2\rvert = \lvert z+i\rvert$	(h) $\lvert z-2\rvert = \lvert z^*+i\rvert$	(i) $\lvert z\rvert = 2\lvert z-2\rvert$
(j) $\arg(z) = \pi/2$	(k) $\arg(z^*) = \pi/4$	(l) $\arg(z) = \lvert z\rvert\,,$

where z^* is the complex conjugate of z.

F8. Describe the curve in the Argand diagram whose equation is

$$|z - 2 - i| = 6\,.$$

As z moves around this curve, what are the loci of

$$u = z + 5 - 8i \qquad \text{and} \qquad v = iz + 2\,.$$

(Note this problem does not require any calculations!)

F9. Show that $z_1 z_2^*$ is invariant under a rotation of z_1 and z_2 about the origin in the Argand diagram.

F10. Show that

$$|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)\,,$$

where a and b are complex numbers. Interpret this result geometrically.

F11. Define u and v to be the real and imaginary parts, respectively, of the complex function $w = 1/z$ with $z = x + iy$. Show that the contours of $u = A =$ const. and $v = B =$ const. (where A and B are real) in the complex plane are circles with centres on the x and y axes respectively, whose radii depend on A and B. Find the radii and centres of the circles and show that they intersect each other at right angles.

F12. Express:

i. $\sin 3\theta$ in terms of powers of $\sin\theta$; and

ii. $\sin^5\theta$ in terms of $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$ etc.

F13. Evaluate the following sums:

$$\text{(a)}\ \sum_{n=1}^{5}\cos n\theta \qquad \text{(b)}\ \sum_{n=1}^{5}\sin n\theta \qquad \text{(c)}\ \sum_{n=1}^{5} r^n\cos n\theta\,.$$

F14. Write the following in the form $a + ib$ where a and b are real:

(a) $e^{-i\pi/2}$ (b) $e^{-i\pi}$ (c) $e^{i\pi/4}$

(d) e^{1+i} (e) $\exp(2e^{i\pi/4})$ (f) $\exp(re^{i\theta})\,.$

(Note that r and θ are real.)

F15. Find all solutions of the following equations and indicate their location on the Argand diagram:

(a) $z^3 = -1$ (b) $z^4 = 1$ (c) $z^2 = i$ (d) $z^3 = -i\,.$

F16. Show that the roots of the equation

$$z^{2n} - 2bz^n + c = 0$$

will, for general complex values of b and c and integral values of n, lie on two circles in the Argand diagram. Give a condition on b and c such that the circles coincide.
Find the largest value of $|z_1 - z_2|$ when z_1 and z_2 are roots of $z^6 - 2z^3 + 2 = 0$.

F17. If $\omega^3 = 1$, find all values of $1 + \omega + \omega^2$. Generalise your result to the case $\omega^n = 1$ for n a positive integer.

F18. Find the (natural) logarithms of the following:

(a) -1 (b) i (c) $1+i$ (d) $(x+iy)^*$ (e) $ire^{i\theta}$.

F19. Show that the displacement from the centre of motion (at $x = 7$) in the simple harmonic motion

$$x = 7 + 24\cos 3t + 7\sin 3t$$

is given by the real part of the complex quantity $X = (24 - 7i)e^{3it}$ and hence obtain the amplitude of the motion. By writing X in the form $Ae^{i\phi}$, where A and ϕ are real, find the time of the first two passages through the centre after $t = 0$ and the distances of the stationary points of the motion from the origin.

G. Hyperbolic functions

G1. Express $\tanh(x+y)$ in terms of $\tanh x$ and $\tanh y$.

G2. Consider the set of points in the x, y plane described by:

i. $(x, y) = (a\cos\theta, b\sin\theta)$; and

ii. $(x, y) = (a\cosh\theta, b\sinh\theta)$,

where a and b are real and fixed and θ is a real parameter. In each case, find the Cartesian equation of the locus and sketch it.

G3. Sketch the graphs of the following functions:

(a) e^x (b) $\cosh x$ (c) $\tanh x$
(d) $\ln x$ (e) $\sinh^{-1} x$ (f) $\tanh^{-1} x$.

G4. Express $\tanh^{-1} x$ as a logarithm and differentiate.

H. Revision of calculus

Basic skills

H1. Calculate the following derivatives:

(a) $\dfrac{d}{dx}(x^2+3)$ (b) $\dfrac{d}{dx}(x^4+2x+6)$ (c) $\dfrac{d}{dx}(ax^n+bx+c)$

(d) $\dfrac{d}{dx}e^x$ (e) $\dfrac{d}{dt}(at+bt^2\sin\theta)$ (f) $\dfrac{d}{dx}(1+x^{2/3})$

(g) $\dfrac{d}{dy}\dfrac{1}{1+y}$ (h) $\dfrac{d}{dx}\ln x\,.$

H2. By writing the trigonometric and hyperbolic functions in terms of exponentials, evaluate the following derivatives:

(a) $\dfrac{d}{dx}\sin x$ (b) $\dfrac{d}{d\theta}\cos\theta$ (c) $\dfrac{d}{dt}\tan t$

(d) $\dfrac{d}{d\omega}\sin(-i\omega t)$ (e) $\dfrac{d}{dx}\sinh x$ (f) $\dfrac{d}{dx}\cosh x$

(g) $\dfrac{d}{dx}\tanh x$ (h) $\dfrac{d}{dx}\tanh 2x\,.$

H3. State whether the following functions are even, odd or neither:

(a) x (b) $(x-a)^2$ (c) $\sin x$ (d) $\sin(\pi/2-x)$

(e) e^x (f) $|x|\cos x$ (g) $\sqrt{x}$ (h) 2

(i) $\ln|(1+x)/(1-x)|\,.$

Main questions

H4. Find dy/dx if $y+e^y\sin y=1/x$ by:

i. expressing x as a function of y and then finding dx/dy; and

ii. differentiating the original expression with respect to x.

(Your answers should agree!)

H5. Sketch the following curves:

(a) $y=(x-3)^3+2x$ (b) $y=\dfrac{x}{1+x^2}$ (c) $y=xe^x$

(d) $y=\dfrac{\ln x}{1+x}$ (e) $y=\dfrac{1}{1-e^x}$ (f) $y=e^x\cos x\,.$

I. Leibnitz's formula

I1. Use Leibnitz's formula to establish that

$$Z_n \equiv \frac{d^n}{dx^n} e^{-x^2/2}$$

is a solution of the differential equation

$$\frac{d^2 Z_n}{dx^2} + x\frac{dZ_n}{dx} + (n+1)Z_n = 0\,.$$

J. Elementary analysis

J1. Explain carefully what is meant by the statement that a function $f(x)$ is:

i. continuous at $x = x_0$; and

ii. differentiable at $x = x_0$.

Sketch graphs of the following functions:

iii. $f(x) = x^2$;

iv. $f(x) = |x|$; and

v. $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

For each of these functions, locate any discontinuities and find the range(s) of x for which they are differentiable.

K. Limits

K1. Investigate the limits, as $n \to \infty$ of the following sequences:

(a) $\dfrac{5^{n+2} - 7^{n+2}}{5^n - 7^n}$ (b) $\dfrac{n + (-1)^n}{n - (-1)^n}$ (c) $\dfrac{n + (-2)^n}{n - (-2)^n}$ (d) $\sqrt{n+1} - \sqrt{n}\,.$

K2. Investigate the limits as $x \to 0_+$ (i.e. x approaches 0 from above) and the limits as $x \to \infty$ of the following functions (where α is real and > 0):

(a) $x^\alpha \ln x$ (b) $x^{-\alpha} \ln x$ (c) $x^\alpha e^{-x}$

(d) $x^{-\alpha} e^x$ (e) $\dfrac{\sin \alpha x}{x}$ (f) $x\cos(\alpha/x)\,.$

L. Convergence of series

L1. Determine which of the series $S = \sum_{n=1}^{\infty} u_n$ are convergent, where:

$$\text{(a)}\quad u_n = \sqrt{n+1} - \sqrt{n} \quad \text{(b)}\quad u_n = \frac{n^2+1}{3n^2+4} \quad \text{(c)}\quad u_n = \frac{n^{10}}{2^n}$$

$$\text{(d)}\quad u_n = \frac{n^{10}}{n!} \quad \text{(e)}\quad u_n = \frac{n!}{10^n} \quad \text{(f)}\quad u_n = \frac{5 \times 8 \times 11 \times \cdots \times (3n+2)}{1 \times 5 \times 9 \times \cdots \times (4n-3)}\,.$$

L2. Determine which of the following series $S = \sum_{n=1}^{\infty} u_n$ are absolutely convergent, and which are conditionally convergent, where:

$$\text{(a)}\quad u_n = \frac{(-1)^{n+1}}{\sqrt{n}} \quad \text{(b)}\quad u_n = (-1)^{n+1}\left(\frac{2n+5}{3n+1}\right)^n \quad \text{(c)}\quad u_n = \frac{(-1)^{n+1}}{(2n-1)^2}\,.$$

M. Power series

M1. Find the first four terms in the Taylor expansion of $\sin x$ about $x = \pi/6$. Hence find an approximate value for $\sin 31°$. Justify the precision of your answer with Taylor's theorem.

M2. Find the first three non-zero terms in the Taylor series expansion about $x = 0$ of:

$$\text{(a)}\quad \sin^{-1} x \qquad \text{(b)}\quad \tan x \qquad \text{(c)}\quad \frac{1}{\sqrt{1+x}}\,.$$

M3. Show that for $|x| \leq 1$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,.$$

i. Using the result $\tan^{-1} 1 = \pi/4$, how many terms of the series are needed to calculate π to 10 places of decimals?

ii. Show that $\pi/4 = \tan^{-1}(1/2) + \tan^{-1}(1/3)$ and deduce another series for π. How many terms of this series are needed to calculate π to 10 decimal places?

iii. Show that $\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$ and deduce yet another series for π. How many terms of this last series are needed to calculate π to 10 decimal places?

N. Approximation

N1. Approximate the following functions in the limit $|x| \ll 1$. In each case give the leading term and the order of the error:

$$\text{(a)} \quad \frac{x^3 + x}{x + 2} \qquad\qquad \text{(b)} \quad \frac{\cos x - 1}{x^2}\,.$$

N2. Approximate the following function in the limit $x \gg 1$. Give the leading term and the order of the error:

$$\frac{1 + 2x + 2x^2}{3x + 3}\,.$$

N3. Use the binomial expansion (or Taylor's theorem) to show that

$$(x^3 + x^2 + 1)^{1/3} - (x^2 + x)^{1/2} = -\frac{1}{6} + \frac{1}{72x} + O\left(\frac{1}{x^2}\right) \text{ as } x \to \infty\,.$$

N4. Show that the volume V and surface area A of a cuboid with sides of length l, $l + 1$ and $l + 2$ are such that

$$V^{1/3} - \left(\frac{A}{6}\right)^{1/2} = -\frac{1}{6l} + O\left(\frac{1}{l^2}\right) \text{ as } l \to \infty\,.$$

O. Newton-Raphson root finding

O1. a)Write down the Taylor series for a function f evaluated at $x+h$ in terms of $f(x)$ and its derivatives evaluated at x. Use this result to show that if x_0 is an approximate solution of the equation $f(x) = 0$, then a better approximation is given, in general, by $x = x_1$ where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$

b) Sketch the graph of $g(x) = x^3 - 3x^2 + 2$.
Use the above formula with an initial guess $x_0 = 2.5$ to obtain an improved estimate, x_1 of the highest root of the equation $g(x) = 0$. Apply the method a second time to get a further estimate x_2 (you may leave your answer in rational form).
Give a sketch showing the progress of the iterations and demonstrate that the sequence $x_0,\ x_1,\ x_2,\ x_3\ \ldots$ will converge to the root.

c)To which root (if any) do you think the Newton Raphson method will converge if:

i) $x_0 = 1.5$

ii) $x_0 = 1.9$

iii) $x_0 = 2.0$

Justify your answers with sketches.
(You may wish to check your answers using MatLab.)

Answers to basic-skills questions

A1. $(-6, -9)$

A2. B and C are closest and are displaced by $\vec{BC} = (-3, 1, 3)$

B1. 2

B2. i. $\cos^{-1}(8/7\sqrt{6}) \approx 62.2°$; ii. $(-2, 0, 1)$ for example, at angle $\cos^{-1}(-\sqrt{5/14}) \approx 126.7°$; iii. $(-1, 0, 3)$ for example, at angle $\cos^{-1}(\sqrt{10/21}) \approx 46.36°$

C1. i. $(5, 8, -6)$; ii. $(-5, -8, 6)$; iii. $(-23, 32, 19)$; iv. $(26, -35, -25)$ compared to $(-39, -81, 53)$; v. 43; vi. $(-39, -81, 53)$

E1. $x = r\cos\phi$ and $y = r\sin\phi$

E2. $r = 2\cos\phi$

F1. (a) $0, -1$; (b) $1, 0$; (c) $0, 1$; (d) $0, -1$; (e) $-1, 0$; (f) $-3/29, 7/29$; (g) $-1, 0$;
(h) $[R\cos\omega t + X\sin\omega t]/(R^2 + X^2), [R\sin\omega t - X\cos\omega t]/(R^2 + X^2)$ where $X \equiv \omega L - 1/(\omega C)$

F2. $z_1 z_2 = 2 + 11i$

F3. (a) $2e^{i\pi/3}$; (b) $\sqrt{2}e^{i3\pi/4}$; (c) $\sqrt{10/3}e^{i\theta}$ where $\theta = \pi + \tan^{-1}(1/3) \approx 198.4°$;
(d) $(R^2 + X^2)^{-1/2}\exp[i(\omega t - \tan^{-1}(X/R))]$ where $X \equiv \omega L - 1/(\omega C)$

F4. (a) $x^2 - y^2, 2xy$; (b) $-y, x$; (c) $x - y, x + y$; (d) $x^3 - 3xy^2 - x^2 + y^2, 3x^2y - y^3 - 2xy$

H1. (a) $2x$; (b) $4x^3 + 2$; (c) $anx^{n-1} + b$; (d) e^x; (e) $a + 2bt\sin\theta$; (f) $2x^{-1/3}/3$;
(g) $-(1 + y)^{-2}$; (h) $1/x$

H2. (a) $\cos x$; (b) $-\sin\theta$; (c) $\sec^2 t$; (d) $-it\cos(-i\omega t)$; (e) $\cosh x$; (f) $\sinh x$; (g) $\mathrm{sech}^2 x$;
(h) $2\mathrm{sech}^2 2x$

H3. (a) odd; (b) neither for $a \neq 0$; (c) odd; (d) even; (e) neither; (f) even; (g) neither;
(h) even; (i) odd