Mathematical Methods III (B Course): Example Sheet 1 Easter 2022

Corrections and suggestions should be emailed to `rlj22@cam.ac.uk`
Questions marked (*) should be attempted only if time allows.

Matrices, determinants, inverses

1. Let $\mathbf{e}_1$ represent the vector "go 1 mile East" and let $\mathbf{e}_2$ represent the vector "go 1 mile North". Let $\mathbf{x}$ be the vector

$$\mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2.$$

Let a new basis be

$$\mathbf{f}_1 = \mathbf{e}_1, \qquad \mathbf{f}_2 = \text{ "go 2 miles North-East".}$$

Write $\mathbf{x}$ in the new basis and identify its co-ordinates.

2. Consider two matrices:

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 & 2 & b \end{pmatrix}.$$

(a) Show that $\mathbf{v}\,\mathbf{u} = 2 + 2b$.
(b) Evaluate the 3×3 matrix $\mathbf{u}\,\mathbf{v}$.

3. The matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

Some matrix products might be

$$\mathbf{A^2}, \quad \mathbf{AB}, \quad \mathbf{AC}, \quad \mathbf{CA}, \quad \mathbf{B^2}, \quad \mathbf{BC}, \quad \mathbf{CB}, \quad \mathbf{C^2}$$

State which of these products exist (that is, which ones are consistent with the rules of matrix multiplication). Evaluate the products that do exist.

4. Let $\mathbf{D}$ be an $(N \times M)$ matrix.
(a) For what values of N and M does $\mathbf{D}\mathbf{D}^T$ exist?
(b) What about $\mathbf{D}^T\mathbf{D}$?

5. Let

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Calculate $\mathbf{M}^2$, $\mathbf{M}\mathbf{M}^T$, $\mathbf{M}^T\mathbf{M}$.

6. Define a new sort of multiplication of matrices *(bullet multiplication)* by

$$(\mathbf{A} \bullet \mathbf{B})_{ij} = \sum_k (\mathbf{A})_{ik} (\mathbf{B})_{jk}.$$

Express the matrix $\mathbf{A} \bullet \mathbf{B}$ as an ordinary product of matrices. Show using suffix notation that bullet multiplication is not associative, that is

$$(\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C} \neq \mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C}).$$

7. Prove the following results:

(i) The trace of the product of a symmetric and an antisymmetric matrix is zero.

(ii) If $\mathbf{A}$ is antisymmetric, then $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$ for any column vector $\mathbf{x}$.

8. Two matrices $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{x}$ have elements a_{ij} , b_{ij} and x_i respectively. (Note: this is not the same convention used in the lecture notes, it is important that you become familiar with different notations.)

In this notation, what are the elements of $\mathbf{A}^T$?

In suffix notation, $\sum_k a_{ik} b_{kj}$ is an element of the matrix $\mathbf{AB}$. Write down the matrices (or vectors) whose elements are

(i) $\sum_{j,k} a_{ij} b_{jk} x_k$,

(ii) $\sum_j x_j b_{ij}$,

(iii) $\sum_{i,j} a_{ij} b_{kj} x_i$,

(iv) $\sum_{j,k} a_{ij} a_{kj} a_{km}$.

9. Produce specific examples to show that

(i) The product of two non-zero matrices can be zero;

(ii) $\mathbf{AB}$ can be zero even if $\mathbf{BA}$ is non-zero;

(iii) The product of two non-zero symmetric matrices can be anti-symmetric.

10. For any square matrix $\mathbf{M}$, the matrix $\exp \mathbf{M}$ is defined by

$$\exp \mathbf{M} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{M}^n}{n!}.$$

For the matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and any real number θ, show that

$$\exp \theta \mathbf{M} = \mathbf{I} + \mathbf{M} \sin\theta + \mathbf{M}^2 (1 - \cos\theta),$$

and hence show that

$$\exp \theta_1 \mathbf{M} \ \exp \theta_2 \mathbf{M} = \exp(\theta_1 + \theta_2) \mathbf{M}.$$

Show also that $(\exp \theta \mathbf{M})(\exp \theta \mathbf{M})^T = \mathbf{I}$. Without detailed calculations, explain whether this result will hold for general $\mathbf{M}$, or whether $\mathbf{M}$ here is of special type (which you should specify).

11. In this question, $\mathbf{a} \times \mathbf{b}$ indicates the vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Let $\mathbf{b} = (b_1, b_2, b_3)$ be a fixed vector. Find the (3×3) matrix $\mathbf{B}$ which satisfies $\mathbf{B}\,\mathbf{x} = \mathbf{b} \times \mathbf{x}$ for any vector $\mathbf{x}$. (That is, find the elements of $\mathbf{B}$ in terms of the components of $\mathbf{b}$).

By considering $\mathbf{B^2x}$, verify the formula $\mathbf{b} \times (\mathbf{b} \times \mathbf{x}) = (\mathbf{b}.\,\mathbf{x})\,\mathbf{b} - (\mathbf{b}.\,\mathbf{b})\,\mathbf{x}$.

12. For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, compute: $\det \mathbf{A}$, $\det \mathbf{B}$, $\det \mathbf{AB}$, and $\det(\mathbf{A}^{-1})$

Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ and $\det \mathbf{A^{-1}} = 1/\det \mathbf{A}$.

13. Calculate the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

as $A_{11}C_{1,1} + A_{12}C_{1,2} + A_{13}C_{1,3}$ where $C_{i,j}$ is the relevant cofactor.

Compute the same determinant by using row operations to convert the determinant to a form with 0's below the diagonal. (See Example 3 in the notes).

Verify that $\det(\mathbf{A}^T) = \det \mathbf{A}$.

14. What can be said about the columns of a matrix whose determinant is zero? For which value of a are the vectors $(1, 0, 1, 0)$, $(2, 1, 3, 2)$, $(4, 0, 1, 3)$ and $(2, 0, 3, a)$ linearly dependent?

15. In the theory of superconductivity, the energy E satisfies an equation of the form

$$\begin{vmatrix} a-E & -b & -b & -b & -b \\ -b & a-E & -b & -b & -b \\ -b & -b & a-E & -b & -b \\ -b & -b & -b & a-E & -b \\ -b & -b & -b & -b & a-E \end{vmatrix} = 0\,.$$

Show by solving this equation that there is one state with energy $E = a - 4b$ and all the others have energy $E = a + b$.

(Hint: it is not recommended to compute the determinant directly from its definition.)

Systems of linear equations

16. Write the set of equations

$$\begin{aligned} p+z &= 2 \\ 2p+y+w &= 4 \\ y+3w &= 3 \\ 2z+p+3y &= 6\,, \end{aligned}$$

in matrix form $\mathbf{A}\,\mathbf{x}=\mathbf{d}$, where

$$\mathbf{x}=\begin{pmatrix} p \\ y \\ z \\ w \end{pmatrix}, \qquad \mathbf{d}=\begin{pmatrix} 2 \\ 4 \\ 3 \\ 6 \end{pmatrix},$$

(You are not requested to solve the equations.)

17. Solve the set of equations

$$\begin{pmatrix} 6 & 7 & 3 & -1 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 19 \\ 4 \\ 8 \end{pmatrix}$$

for x, y, z, w.

You should find that matrix equations in this form are straightforward to solve.

18. Solve the equations

$$3x+2y=9, \qquad x+3y=17,$$

by the method of Gaussian elimination. (This means: multiply the first equation by a suitable constant and then subtract it from the second equation, to obtain an equation where x does not appear. Write the resulting two equations in matrix form and compare with the previous question. Hence obtain values for y and x.)

Note: This method can be generalised to matrices of any size, the idea is to add up multiples of different equations until we end up with a set of equations in the form of the previous question.

Solve the same equations by writing them in the form $\mathbf{Ax}=\mathbf{b}$ and computing the inverse of $\mathbf{A}$.

19. Compute the inverse of $\mathbf{A}$ and hence solve the matrix equation $\mathbf{Ax}=\mathbf{y}$, where

$$\mathbf{A}=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \qquad \mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{y}=\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ be the solutions of the above equation corresponding to the three cases $\mathbf{y}=(1,0,0)$, $\mathbf{y}=(0,1,0)$, $\mathbf{y}=(0,0,1)$ respectively. What do you notice about the matrix whose columns are $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$?

20. Suppose that two $(N \times N)$ matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{AB} = \mathbf{I}$. The N rows of $\mathbf{A}$ are denoted $\mathbf{a}_i$ for $1 \le i \le N$ and the N columns of $\mathbf{B}$ are denoted by $\mathbf{b}_j$ for $1 \le j \le N$. What can be said about the scalar products $\mathbf{a}_i \cdot \mathbf{b}_j$?

Show that in the case $N = 3$, these relationships are satisfied by the vectors $\mathbf{a}_i$ and

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]} \quad \mathbf{b}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]} \quad \mathbf{b}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]}$$

where $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ denotes the scalar triple product.

21. Cramer's rule for solving the equation $\mathbf{Ax} = \mathbf{y}$ is

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}},$$

where $\mathbf{A}_i$ is the matrix obtained by replacing the *i-th* column of $\mathbf{A}$ by column vector $\mathbf{y}$. Use the cofactor expression for the inverse of a matrix to derive Cramer's rule in the 3×3 case, commenting on the cases where there is no solution and where the solution is not unique.

22. Write the following equations in the form $\mathbf{Ax} = \mathbf{y}$:

$$\begin{aligned} x + y + az &= 1 \\ 3x + 4y + (2 + 3a)z &= 5 \\ -x + y + z &= b \end{aligned}$$

Calculate $|\mathbf{A}|$. Under what circumstances is $|\mathbf{A}| = 0$? For which values of a, b do the equations have a unique solution?

For the cases where a unique solution does not exist, determine the values of b for which there are no solutions. Hence work out the values for which there are many solutions.

Hint: if there are no solutions then you should be able to generate an inconsistent equation.

23. Find a 3×3 matrix $\mathbf{A}$ such that the equation

$$x^2 + 4y^2 + 9z^2 + 4xy - 6xz = 1$$

can be written in the matrix form

$$\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1.$$

Linear transformations, eigenvalues, eigenvectors

24. Find the rotation matrix $\mathbf{R}$ such that $f(\mathbf{x}) = \mathbf{R}\,\mathbf{x}$ represents an anticlockwise rotation about the z-axis through an angle of $\pi/2$ radians.

25. In this question, you should use the fact that a matrix describes a linear mapping, but should not be necessary to write down any matrices.

The effect of a 3×3 *reflection matrix* is to reflect vectors in a fixed plane; in other words, the component of the vector normal to the plane is reversed and the component parallel to the plane is unchanged.

What do you think the eigenvalues of a reflection matrix are? What is the geometrical significance of the corresponding eigenvectors. Are they orthogonal?

26. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{M} = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Verify that: (i) the eigenvalues are real; (ii) the sum of the eigenvalues is equal to trace $\mathbf{M}$; (iii) the product of the eigenvalues is equal to det $\mathbf{M}$; (iv) the eigenvectors are orthogonal.

27. Find the eigenvalues and eigenvectors ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$), of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

Let $\mathbf{y} = (6, -3, 0)^T$. Observe that the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ can be used as an orthonormal basis. By taking dot products with each of the eigenvectors in turn, find the co-ordinates of $\mathbf{y}$ in this basis.

[Recall: the co-ordinates (p_1, p_2, p_3) of $\mathbf{y}$ obey $\mathbf{y} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$.]

Let (q_1, q_2, q_3) be the co-ordinates of some vector $\mathbf{x}$, so $\mathbf{x} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$. Use your expressions for $\mathbf{x}$ and $\mathbf{y}$ to solve the equation $\mathbf{A}\mathbf{x} = \mathbf{y}$. Is the solution unique?

28. The column vector $\mathbf{e}$ is n-dimensional and has unit length. The $n \times n$ matrix $\mathbf{A}$ is given by $\mathbf{A} = \lambda\mathbf{e}\mathbf{e}^T$, where λ is a real number. Show that $\mathbf{A}$ has eigenvector $\mathbf{e}$ with eigenvalue λ and that all other $n-1$ eigenvalues of $\mathbf{A}$ are zero.

The real symmetric $n \times n$ matrix $\mathbf{B}$ has orthonormal eigenvectors $\mathbf{e}_a,\ a = 1 \ldots n$ with eigenvalues $\lambda_a,\ a = 1 \ldots n$. Show that $\mathbf{B}$ can be written as

$$\mathbf{B} \;=\; \lambda_1\mathbf{e}_1\mathbf{e}_1^T + \lambda_2\mathbf{e}_2\mathbf{e}_2^T + \ldots + \lambda_n\mathbf{e}_n\mathbf{e}_n^T.$$

(*This is how examiners construct examples of matrices for exams so that the answers are nice numbers.*)

29. The equation $x^2 + 16xy - 11y^2 = 1$ describes a pair of curves in the (x, y) plane. Write the equation in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} = 1$, where $\mathbf{A}$ is a 2×2 matrix. Find the eigenvalues and eigenvectors of $\mathbf{A}$.

Define a new co-ordinate system whose axes are aligned with the eigenvectors of $\mathbf{A}$. Write the equation that describes the curves, in the new co-ordinate system.

30. Find the the stationary points of the function $f(x, y) = x^3 + xy + y^2$. For each point, use the eigenvalues of the Hessian to determine their type (minimum, saddle, etc).

31*. Verify that the following matrix is orthogonal:

$$\mathbf{O} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \quad .$$

The linear map $f(\mathbf{x}) = \mathbf{O}\mathbf{x}$ is a rotation about some axis. If $\mathbf{x}$ is aligned with this axis, what can you say about $f(\mathbf{x})$? Explain why 1 must be an eigenvalue of $\mathbf{O}$ and find the corresponding axis of rotation.

Harder part: By considering geometrically the effect of the matrix on vectors that lie in the 2D subspace orthogonal to the axis, deduce that such vectors rotate according to the orthogonal matrix (defined to act in the 2D subspace)

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ,$$

where θ is the angle of rotation. By considering the characteristic polynomial of $\mathbf{R}$, show that its eigenvalues are $e^{i\theta}, e^{-i\theta}$. Hence deduce that that trace $\mathbf{O} = 1 + 2\cos\theta$. and find the angle θ associated with for the rotation matrix $\mathbf{O}$.