NST1A: Mathematics I (Course B)

Recaps of Lectures

9:00, Tuesday, Thursday Saturday, Michaelmas 2020

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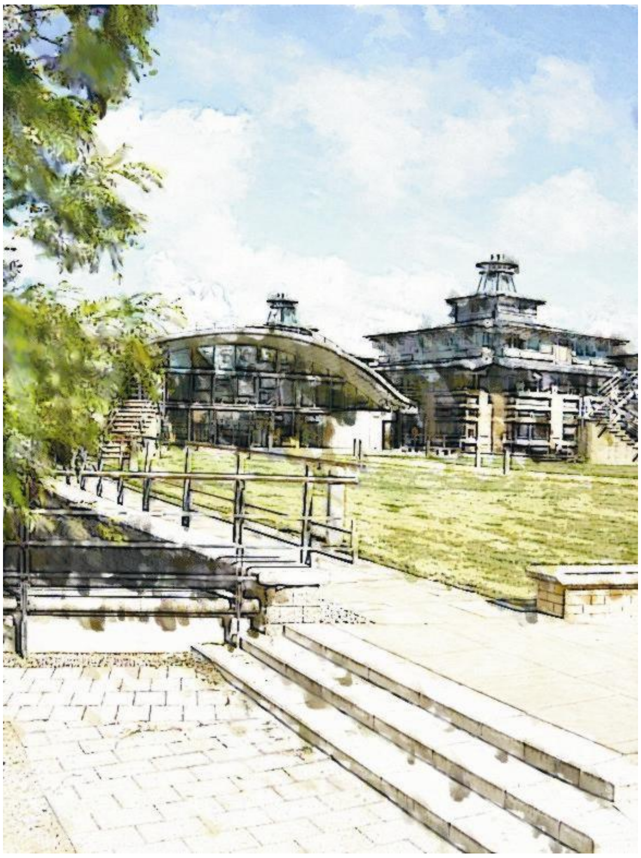


1. Vectors

Recap of Lecture 1: Vectors

Scalar: Value; generally in $\mathbb{R}$, but could be in $\mathbb{C}$

Vector: Magnitude and direction

Displacement vector: relative position

Position vector: position relative to origin

This part of the course concentrates on *physical* space

Euclidian space $\Rightarrow$ can use Cartesian coordinates

3D Euclidian space is $\mathbb{R}^3$ - three real vector components

[There will be some discussion of other bases (*e.g.* $\mathbb{R}^n$ eigenvector basis) in Easter term, but more (*e.g.* Hilbert space $\mathbb{C}^n$) in Part IB.]

$\mathbf{u} \equiv \boldsymbol{u} \equiv \underset{\sim}{u} \equiv \underline{u} \equiv \vec{u}$

Magnitude: $|\mathbf{u}|$

Unit vector: $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$

Vector addition commutative: $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$.

Vector addition associative: $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$

Vector subtraction: $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})=(-\mathbf{b})+\mathbf{a}$

Multiplication by scalar distributive: $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$

$|\lambda \mathbf{a}| = |\lambda|\,|\mathbf{a}|$ ***Notes (p.5) were missing the absolute value on λ

Kinematics: mathematics of motion:

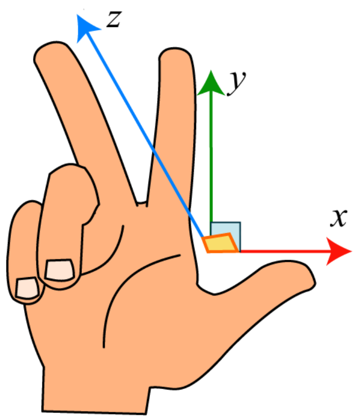
$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\mathbf{u}(t+\delta t) - \mathbf{u}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\delta \mathbf{u}}{\delta t} \equiv \frac{d\mathbf{u}}{dt} \equiv \dot{\mathbf{u}} = \ddot{\mathbf{r}} \equiv \frac{d^2\mathbf{r}}{dt^2}$$

Coordinate axes: effect vector components – the ***basis*** of the vector space

Need to ***span*** the space

Coordinate system has axes and origin.

Unit vectors for Cartesian coordinates: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ or $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ or $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$



Right-handed system

Now Lecture 2: Vectors

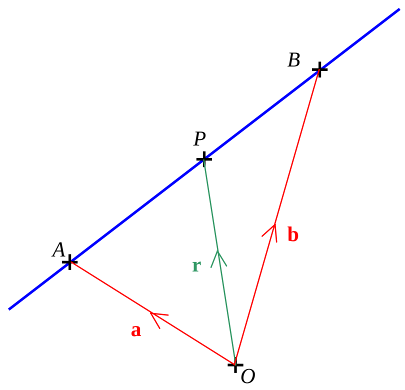
Recap of Lecture 2: Vectors

Basis vectors need not be orthogonal (and need not be unit length), but this can make vector algebra and vector calculus harder.

Vector length: $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

Vector equation of line described by $\mathbf{r}$ through points with position vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}), \quad \lambda \in \mathbb{R}.$$



$$\mathbf{r} = \mathbf{a} + \hat{\lambda}\hat{\mathbf{t}}.$$

Component equation of a straight line: $\dfrac{x - a_x}{b_x - a_x} = \dfrac{y - a_y}{b_y - a_y} = \dfrac{z - a_z}{b_z - a_z}$

> What happens if, for example, $a_x = b_x$?
>
> What if $a_x = b_x$, $a_y = b_y$ and $a_z = b_z$?

Scalar product, ***dot product*** or ***inner product***:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

For ***orthonormal*** basis (basis vectors orthogonal and unit length)

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

(Ideas can be extended to higher dimensional or *infinite dimensional* spaces)

Scalar product commutative: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

Scalar product distributive: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

Scalar product of vector with itself: $\mathbf{a}^2\equiv\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$

Scalar product zero if vectors orthogonal

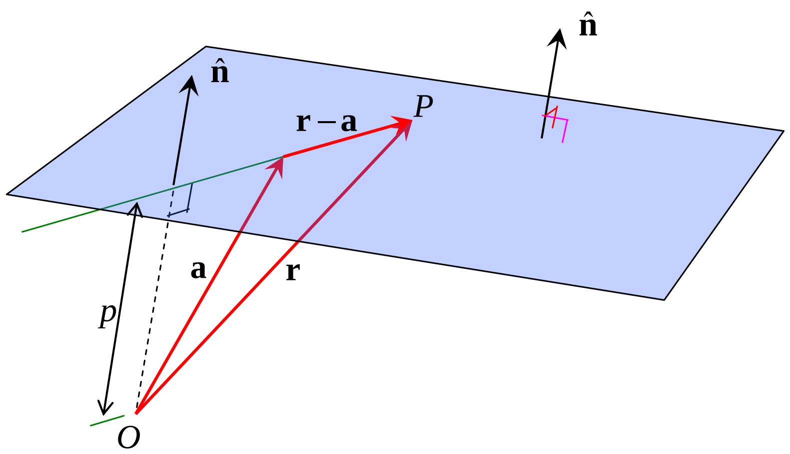
If scalar product zero then either vectors orthogonal or at least one has zero length

Scalar product, ***dot product*** or ***inner product***:

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=a_xb_x+a_yb_y+a_zb_z\ .$$

Orthonormal: Unit vectors normal to each other.

Vector equation of a plane: $(\mathbf{r}-\mathbf{a})\cdot\hat{\mathbf{n}}=0$ or $\mathbf{r}\cdot\hat{\mathbf{n}}=p$



For $(\mathbf{r}-\mathbf{a})\cdot\hat{\mathbf{n}}=0$, then $(\mathbf{r}-\mathbf{a})\cdot(\lambda\hat{\mathbf{n}})=0$, $\lambda\in\mathbb{R}$, but for $\mathbf{r}\cdot\hat{\mathbf{n}}=p$ need to stick with unit normal and agree *which* unit normal.

Direction cosines: l,m,n in $\mathbf{r}\cdot\hat{\mathbf{n}}=lx+my+nz=p$

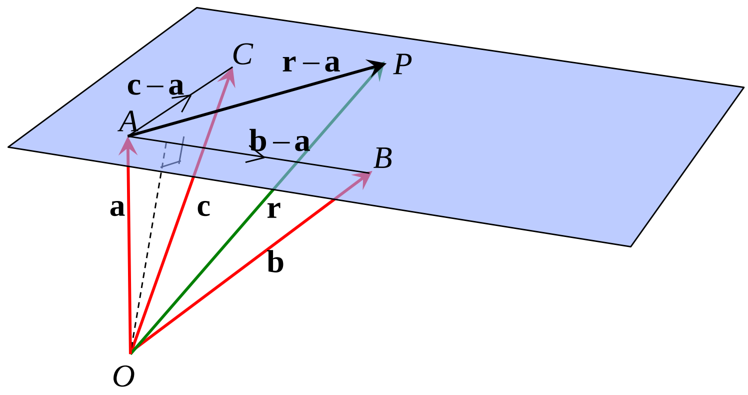
Note: $l^2+m^2+n^2=1$ since $|\hat{\mathbf{n}}|=1$

$\Rightarrow\ l=\cos\theta_x,\ m=\cos\theta_y,\ n=\cos\theta_z$

Now Lecture 3: Vectors

Recap of Lecture 3: Vectors

Equation of a plane: $\mathbf{r}-\mathbf{a}=\mu(\mathbf{b}-\mathbf{a})+\nu(\mathbf{c}-\mathbf{a}),\ \mu,\nu\in\mathbb{R}$



Can use $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ as basis for points on the plane; in that 2D basis, $\mathbf{r}-\mathbf{a}=(\mu,\nu)$

. Vectors *The equation of a plane*

Example 10: Visibility over a wall

Can an object at the point $(2,3,3/2)$ be seen by an observer at $(4,5,1)$ when there is an intervening wall with top given by the line

C D

$$\mathbf{r}=(1,2,1)+\lambda(1,-1,-1) \quad (32)$$

A

[the z axis is the upward vertical]?

$\underset{\sim}{r}=\underset{\sim}{a}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})$

Answer on Moodle

Assume z is upwards

+ D

ABC define plane

$\underset{\sim}{a}=(1,2,1)$

$\underset{\sim}{b}-\underset{\sim}{a}=(1,-1,-1)$

$\underset{\sim}{c}=(4,5,1)$

$\underset{\sim}{d}=(2,3,\tfrac{3}{2})$

Equation of plane

$\underset{\sim}{s}=\underset{\sim}{a}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})+\mu(\underset{\sim}{c}-\underset{\sim}{a})$

$=(1,2,1)+\lambda(1,-1,-1)+\mu(3,3,0)$

$=(x,y,z)$

$\Rightarrow \left.\begin{aligned} x&=1+\lambda+3\mu\\ y&=2-\lambda+3\mu\\ z&=1-\lambda\end{aligned}\right\}$ Find λ & μ that match x & y of D

$x=1+\lambda+3\mu=2 \Rightarrow \lambda+3\mu=1$

$y=2-\lambda+3\mu=3 \Rightarrow -\lambda+3\mu=1$

$6\mu=2 \Rightarrow \mu=\tfrac{1}{3}$

$2\lambda=0 \Rightarrow \lambda=0$

$\Rightarrow$ At $x=1+0+3\times\tfrac{1}{3}=2,\ y=2-0+3\times\tfrac{1}{3}=3$

then $z=1-0=1$

Hence the plane passes through $x=2,\ y=3$ below the point D $(z=\tfrac{3}{2})$.

$\Rightarrow$ The object is visible

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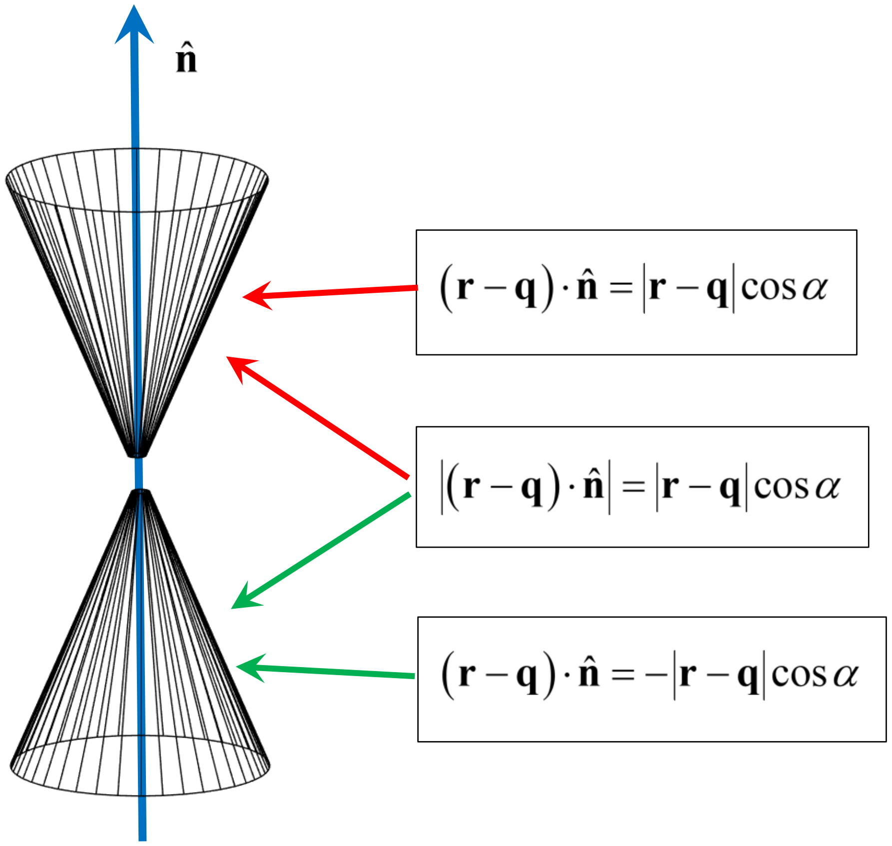
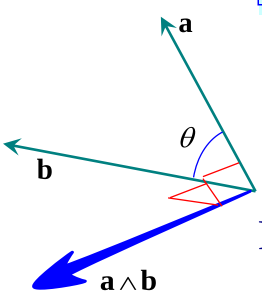
Equation for a sphere: $|\mathbf{r}-\mathbf{a}|=\rho$

Equation for a cylinder: $|\mathbf{r}-(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}|=R$

Equation for a cone: $(\mathbf{r}-\mathbf{q})\cdot\hat{\mathbf{n}}=|\mathbf{r}-\mathbf{q}|\cos\alpha$



Vector or cross product $\mathbf{a}\wedge\mathbf{b}\equiv|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$



The unit normal $\hat{\mathbf{n}}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Its direction is determined by the right-hand rule.

For right-handed orthonormal coordinate system

$$\hat{\mathbf{i}}\wedge\hat{\mathbf{i}}=\hat{\mathbf{j}}\wedge\hat{\mathbf{j}}=\hat{\mathbf{k}}\wedge\hat{\mathbf{k}}=\mathbf{0}. \tag{21}$$

$$\hat{\mathbf{i}}\wedge\hat{\mathbf{j}}=\hat{\mathbf{k}},\ \hat{\mathbf{j}}\wedge\hat{\mathbf{k}}=\hat{\mathbf{i}},\ \hat{\mathbf{k}}\wedge\hat{\mathbf{i}}=\hat{\mathbf{j}} \tag{22}$$

$$\rightarrow\quad \mathbf{a}\wedge\mathbf{b}=(a_yb_z-a_zb_y)\hat{\mathbf{i}}+(a_zb_x-a_xb_z)\hat{\mathbf{j}}+(a_xb_y-a_yb_x)\hat{\mathbf{k}}$$

Vector product anti-commutative: $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$

Vector product distributive: $\mathbf{a}\wedge(\mathbf{b}+\mathbf{c}) = \mathbf{a}\wedge\mathbf{b}+\mathbf{a}\wedge\mathbf{c}$

Vector product not associative: $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) \neq (\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$

Vector product using determinants

- Laplace expansion:

$$\mathbf{a}\wedge\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(a_y b_z - a_z b_y\right) - \hat{\mathbf{j}}\left(a_x b_z - a_z b_x\right) + \hat{\mathbf{k}}\left(a_x b_y - a_y b_x\right)$$

- Rule of Sarrus

$$\left|\begin{array}{ccc|ccc} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z & a_x & a_y & a_z \\ b_x & b_y & b_z & b_x & b_y & b_z \end{array}\right. = \hat{\mathbf{i}}\left(a_y b_z - a_z b_y\right) + \hat{\mathbf{j}}\left(a_z b_x - a_x b_z\right) + \hat{\mathbf{k}}\left(a_x b_y - a_y b_x\right)$$

$$\left|\begin{array}{ccc|ccc} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} & \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z & a_x & a_y & a_z \\ b_x & b_y & b_z & b_x & b_y & b_z \end{array}\right. = \hat{\mathbf{i}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} + \hat{\mathbf{j}}\begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(a_y b_z - a_z b_y\right) + \hat{\mathbf{j}}\left(a_z b_x - a_x b_z\right) + \hat{\mathbf{k}}\left(a_x b_y - a_y b_x\right)$$

Each term has one thing from the first row ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ or $\hat{\mathbf{k}}$), one from the second row (an element of $\mathbf{a}$) and one from the third row (an element of $\mathbf{b}$).

Each term also has one thing from each column: something in the x direction, something in the y direction and something in the z direction.

If the $xyzxyz\cdots$ cyclic order is preserved with things in row order, then the sign is positive. If it is not preserved, then the sign is negative.

For example, $\hat{\mathbf{j}}a_z b_x$ preserves the cyclic order, and so this term is positive. In contrast, $\hat{\mathbf{i}}a_z b_y$ reverses the cyclic order, and so has a negative sign.

As multiplication by scalars is commutative, switching to $a_z\hat{\mathbf{i}}b_x$ may appear to reverse the cyclic order, but as the cyclic order of the rows is also reversed, then it still has a positive sign.

For this part of the course, you do not need to know what a matrix or determinant is: you only need to be able to determine the vector product

In the NST1B maths:

$$\mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \varepsilon_{ijk} a_j b_k$$

Einstein notation implies summation over repeated indices so

$$c_i = \varepsilon_{ijk} a_j b_k \equiv \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k \,.$$

Here,

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

is the Levi-Civita symbol.

Even permutations: 123, 231, 312

Odd permutations: 321, 213, 132

0 permutations $\quad 123 \to 123 \quad \Rightarrow \quad 1$

1 permutation $\quad 123 \to 321 \quad \Rightarrow \quad -1$

2 permutations $\quad 321 \to 312 \quad \Rightarrow \quad 1$

Note: The Levi-Civita symbol is **not** a tensor as it does not obey the tensor transformation rules.

Now Lecture 4: Vectors

Recap of Lecture 4: Vectors

Example 12: Spinning body

A rigid body spins with angular speed ω about an axis $\hat{\mathbf{n}}$ passing through the point Q. Find a vector expression for the velocity of any point in the body.

Answer on moodle.

$\underset{\sim}{p} = \vec{QP} = \underset{\sim}{r} - \underset{\sim}{q}$

Distance from P to axis?

$$\begin{aligned}\rho &= |\underset{\sim}{p}| \sin\theta \\ &= |\underset{\sim}{r} - \underset{\sim}{q}| \sin\theta \\ &= |\hat{\underset{\sim}{n}}||\underset{\sim}{r} - \underset{\sim}{q}| \sin\theta \\ &= |\hat{\underset{\sim}{n}} \times (\underset{\sim}{r} - \underset{\sim}{q})|\end{aligned}$$

Linear speed $|\underset{\sim}{u}| = \omega\rho$

Direction (recall it is circular motion) is perpendicular to $\hat{\underset{\sim}{n}}$ and $\underset{\sim}{p}$

Recall $\hat{\underset{\sim}{n}} \times (\underset{\sim}{r} - \underset{\sim}{q}) = |\hat{\underset{\sim}{n}}||\underset{\sim}{r} - \underset{\sim}{q}| \sin\theta \; \hat{\underset{\sim}{m}}$ where $\hat{\underset{\sim}{m}}$ is perpendicular to $\hat{\underset{\sim}{n}}$ & $\underset{\sim}{r} - \underset{\sim}{q} = \underset{\sim}{p}$

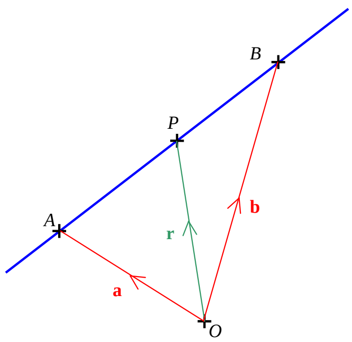
Need to think whether we want $\hat{\underset{\sim}{n}} \times \underset{\sim}{p}$ or $\underset{\sim}{p} \times \hat{\underset{\sim}{n}}$?

As drawn, $\underset{\sim}{u}$ into page so want $\hat{\underset{\sim}{n}} \times \underset{\sim}{p}$

So linear velocity $\underset{\sim}{u} = \omega\rho\, \hat{\underset{\sim}{m}} = \omega |\underset{\sim}{r} - \underset{\sim}{q}| \sin\theta\, \hat{\underset{\sim}{m}}$
$= \omega(\hat{\underset{\sim}{n}} \times (\underset{\sim}{r} - \underset{\sim}{q}))$

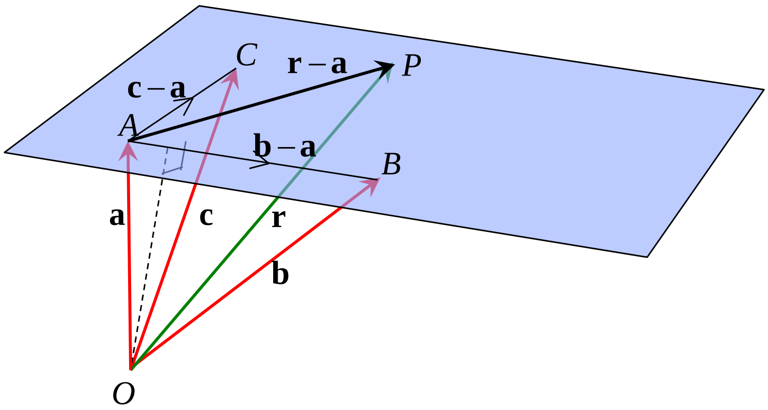
Lines and planes

Vector equation of line passing through points given by $\mathbf{a}$ and $\mathbf{b}$:



$$(\mathbf{r} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a}) = \mathbf{0}$$

$$\Rightarrow \mathbf{r} - \mathbf{a} \text{ is parallel to } \mathbf{b} - \mathbf{a}$$

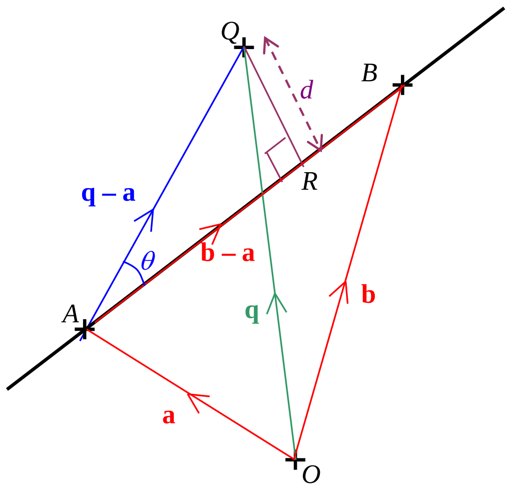


Vector equation of plane through points given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

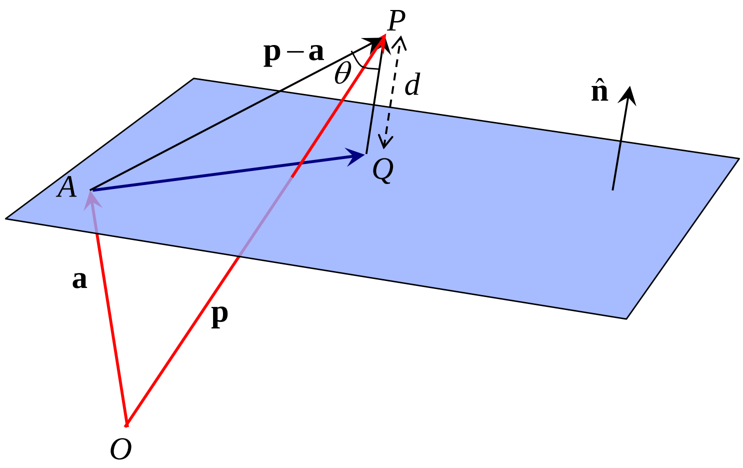
$$(\mathbf{r}-\mathbf{a})\cdot\left[(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a})\right]=0$$

Shortest distances

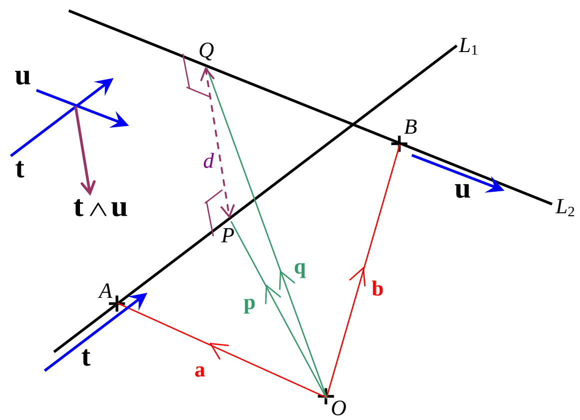
Shortest distance to line: $d=\dfrac{\left|(\mathbf{q}-\mathbf{a})\wedge(\mathbf{b}-\mathbf{a})\right|}{\left|\mathbf{b}-\mathbf{a}\right|}$



Shortest distance to plane: $d = \left|(\mathbf{p}-\mathbf{a})\cdot\hat{\mathbf{n}}\right|$



Shortest distance between two lines $d = \dfrac{\left|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{t}\wedge\mathbf{u})\right|}{\left|\mathbf{t}\wedge\mathbf{u}\right|}$



A line in direction $\mathbf{t}$ passing through a point given by position vector $\mathbf{a}$ intersects with one with direction $\mathbf{u}$ passing through a point given by position vector $\mathbf{b}$ intersects if the scalar triple product vanishes: $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{t}\wedge\mathbf{u})=0$.

Scalar triple product

$$\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}=\mathbf{b}\cdot\mathbf{c}\wedge\mathbf{a}=\mathbf{c}\cdot\mathbf{a}\wedge\mathbf{b}$$
$$=-\mathbf{a}\cdot\mathbf{c}\wedge\mathbf{b}=-\mathbf{b}\cdot\mathbf{a}\wedge\mathbf{c}=-\mathbf{c}\cdot\mathbf{b}\wedge\mathbf{a}$$

cyclic order (123, 231, 312) maintain sign,
anticyclic order (321, 213, 132) swap sign
even permutations retain sign, odd permutations swap sign

$$\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}\equiv\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$$

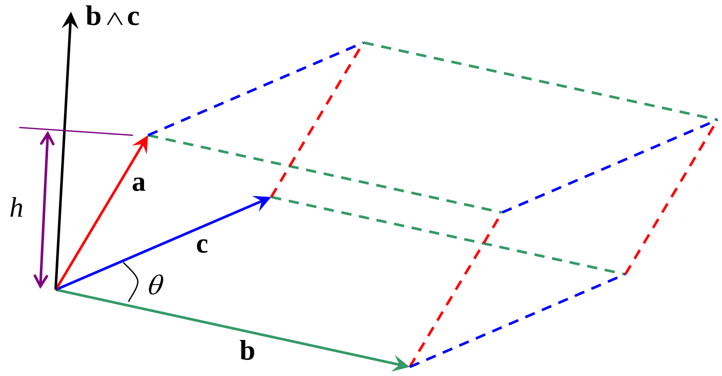
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}&=|\mathbf{a}||\mathbf{b}\wedge\mathbf{c}|\cos\theta_{a(b\wedge c)}\\
&=|\mathbf{a}|\;\big||\mathbf{b}||\mathbf{c}|\sin\theta_{bc}\hat{\mathbf{n}}\big|\;\cos\theta_{a(b\wedge c)}\\
&=|\mathbf{a}||\mathbf{b}||\mathbf{c}|\cos\theta_{a(b\wedge c)}\sin\theta_{bc}\\
&=|\mathbf{a}||\mathbf{b}||\mathbf{c}|\cos\theta_{b(c\wedge a)}\sin\theta_{ca}\\
&=|\mathbf{a}||\mathbf{b}||\mathbf{c}|\cos\theta_{c(a\wedge b)}\sin\theta_{ab}\end{aligned}$$

What is the angle between two vectors?

$$\theta=\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\theta=\sin^{-1}\frac{|\mathbf{a}\wedge\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

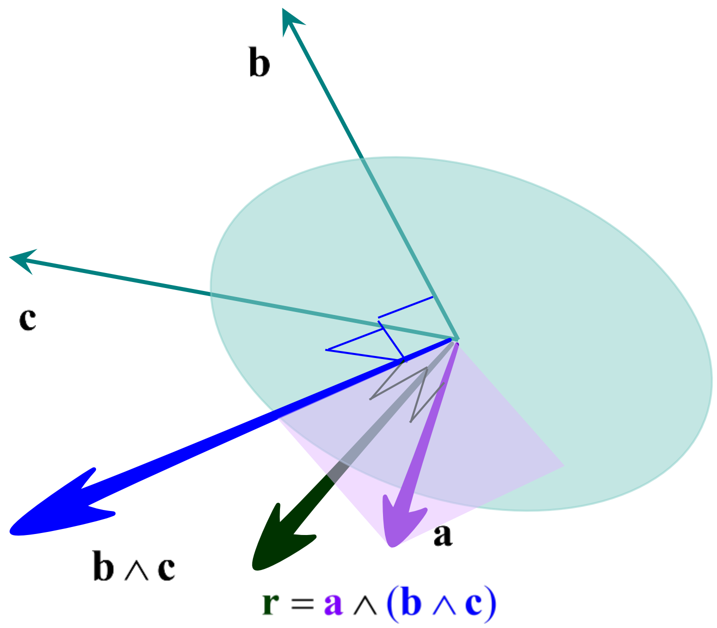
Volume of parallelepiped formed by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $V = |\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}|$



Vector triple product

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

normal to $\mathbf{a}$ *and normal to the normal to both* $\mathbf{b}$ *and* $\mathbf{c}$



$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$$

Einstein: $\mathbf{r} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \equiv \varepsilon_{ijk}\varepsilon_{klm} a_j b_l c_m = r_i$

$$\mathbf{s} = (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} \equiv \varepsilon_{ijk}\varepsilon_{jlm} a_l b_m c_k = s_i$$

identity $$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\boxed{\begin{gathered}\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})\\ = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\end{gathered}}$$

$$\boxed{\begin{gathered}(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}\\ = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a}\end{gathered}}$$

Provided $\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}\neq 0$, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ span 3D space and any other vector can be written as a linear combination of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\begin{aligned}\mathbf{d} &= d_a\mathbf{a} + d_b\mathbf{b} + d_c\mathbf{c}\\ &= \frac{(\mathbf{d}\cdot\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{d}\cdot\mathbf{c}\wedge\mathbf{a})\mathbf{b} + (\mathbf{d}\cdot\mathbf{a}\wedge\mathbf{b})\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}}\end{aligned}$$

$\Rightarrow$
$$d_a = \frac{\mathbf{d}\cdot\mathbf{b}\wedge\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}},$$

$$d_b = \frac{\mathbf{d}\cdot\mathbf{c}\wedge\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}},$$

$$d_c = \frac{\mathbf{d}\cdot\mathbf{a}\wedge\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}}.$$

How much of $\mathbf{a}$ is not in plane of $\mathbf{b}, \mathbf{c}$? $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

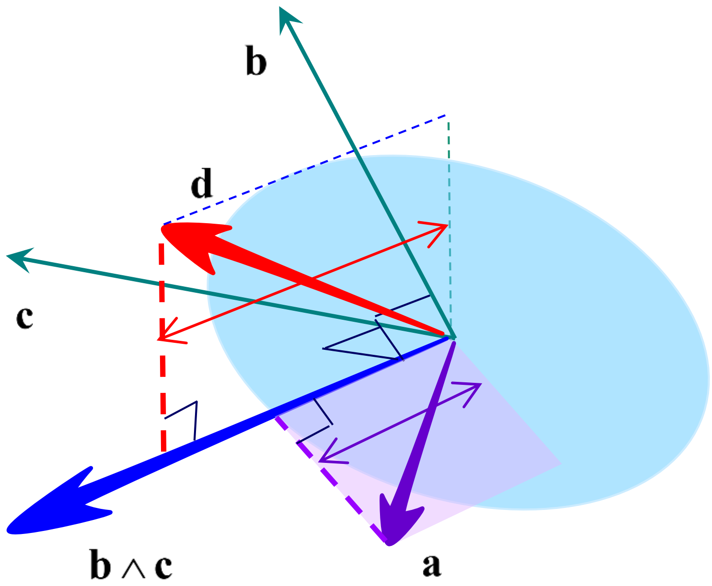
How much of $\mathbf{d}$ is not in plane of $\mathbf{b}, \mathbf{c}$? $\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})$
This **must** be captured by the parts of $\mathbf{a}$ that are orthogonal to plane of $\mathbf{b}, \mathbf{c}$

$$d_a = \frac{\mathbf{d}\cdot\mathbf{b}\wedge\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}}$$

Similarly

$$d_b = \frac{\mathbf{d}\cdot\mathbf{c}\wedge\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}} = \frac{\mathbf{d}\cdot\mathbf{c}\wedge\mathbf{a}}{\mathbf{b}\cdot\mathbf{c}\wedge\mathbf{a}},$$

$$d_c = \frac{\mathbf{d}\cdot\mathbf{a}\wedge\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}} = \frac{\mathbf{d}\cdot\mathbf{a}\wedge\mathbf{b}}{\mathbf{c}\cdot\mathbf{a}\wedge\mathbf{b}}.$$



$$d_a = \frac{\mathbf{d}\cdot\mathbf{b}\wedge\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}}$$

Election of class representative?

"The Faculty Board of Mathematics asked DAMTP to set up a Staff-Student Committee for Mathematics in the Natural Sciences to provide an opportunity for discussion of matters relating to the courses. The Committee has four staff and three student members, the latter being drawn from the A and B courses in Part IA and from the Part IB course."

Now Lecture 5: Vectors

Recap of Lecture 5: Vectors

Units of vectors

Note: if $\mathbf{a},\mathbf{b},\mathbf{c}$ all have the dimensions of length, then

1. $|\mathbf{a}|$, $|\mathbf{b}|$, $|\mathbf{c}|$ all have dimensions of length;
2. $\mathbf{a}\cdot\mathbf{b}$ is a scalar with dimensions of area (length-squared);
3. $\mathbf{b}\wedge\mathbf{c}$ is a vector with dimensions of area (length-squared);
4. $\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}$ is a scalar with dimensions of volume (length-cubed);
5. $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ is a vector with dimensions of volume (length-cubed);
6. Normalising a vector (so it has unit length) removes its dimensions: $\hat{\mathbf{n}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ does not have physical dimensions;
7. $\mathbf{b}\cdot\hat{\mathbf{n}}$ has the same dimensions as $\mathbf{b}$ (*i.e.* physical length).

Vector area

For a plane of area A

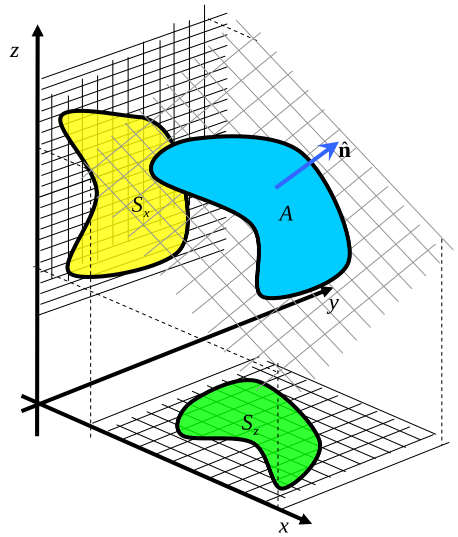
$$\mathbf{S} = A\hat{\mathbf{n}}.$$

Lent term: $\int_S \mathbf{dS} = \int_S \hat{\mathbf{n}}\, dA$

$\rightarrow$ surface integrals $\int_S f(\mathbf{x})\mathbf{dS} = \int_S f(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x})\, dA$

Has dimensions of area A but with a direction

$$\begin{aligned}\mathbf{S} &\equiv S_x\hat{\mathbf{i}} + S_y\hat{\mathbf{j}} + S_z\hat{\mathbf{k}} \\ &= A\left[(\hat{\mathbf{i}}\cdot\hat{\mathbf{n}})\hat{\mathbf{i}} + (\hat{\mathbf{j}}\cdot\hat{\mathbf{n}})\hat{\mathbf{j}} + (\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})\hat{\mathbf{k}}\right] \\ &= A(\cos\theta_x, \cos\theta_y, \cos\theta_z)\end{aligned}$$



Components of $\mathbf{S}$ are the projections of the area onto the planes normal to the axes.

If $\mathbf{S} = A\hat{\mathbf{n}}$ then $|\mathbf{S}| = |A\hat{\mathbf{n}}| = |A||\hat{\mathbf{n}}| = |A|$; if $A \geq 0$ then $|\mathbf{S}| = A$

Vector area of a closed volume is zero: $\mathbf{S} = \mathbf{0}$

For an open surface/shell (there is only one 'side'), then $\mathbf{S} \neq \mathbf{0}$

Vector basis

Basis: a system of vectors used to represent a position in a space.

1. Vectors must be linearly independent

$$\lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2+\ldots+\lambda_N\mathbf{e}_N=\mathbf{0} \quad \Rightarrow \quad \lambda_1=\lambda_2=\ldots=\lambda_N=0. \tag{42}$$

2. Number of vectors must equal the dimensions of the space

Linear independence means $\mathbf{r}+\mathbf{a}=\mathbf{r}$ has the unique solution $\mathbf{a}=\mathbf{0}$;

$\Rightarrow$ a unique set of coefficients $\gamma_1,\gamma_2,\ldots,\gamma_N$ for every point

$$\mathbf{r}=\gamma_1\mathbf{e}_1+\gamma_2\mathbf{e}_2+\ldots+\gamma_N\mathbf{e}_N \tag{43}$$

$$\begin{vmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \vdots\\ \mathbf{e}_N\end{vmatrix}=\begin{vmatrix}e_{1p} & e_{1q} & \cdots & e_{1N}\\ e_{2p} & e_{2q} & \cdots & e_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ e_{Np} & e_{Nq} & \cdots & e_{NN}\end{vmatrix}\neq 0$$

In 3D, $\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3\neq 0$

Orthogonal basis: A basis in which all the basis vectors are orthogonal

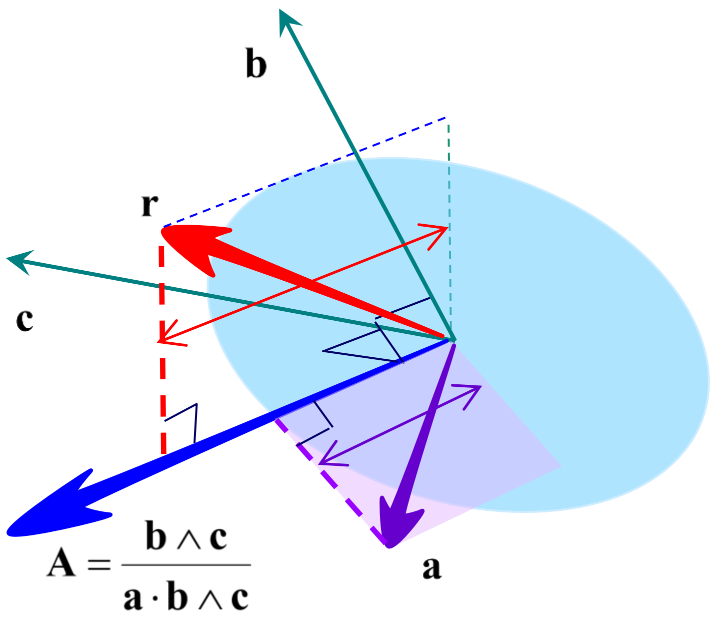
Orthonormal basis: An orthogonal basis in which the basis vectors all have unit length.

Scalar product if orthonormal basis: $\mathbf{a}\cdot\mathbf{b}=\sum_{i=1}^{N}a_ib_i$.

Reciprocal basis – related to the inverse of a matrix comprising the basis vectors. Basis vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ has the reciprocal basis

$$\mathbf{A}\equiv\frac{\mathbf{b}\wedge\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}},\ \mathbf{B}\equiv\frac{\mathbf{c}\wedge\mathbf{a}}{\mathbf{b}\cdot\mathbf{c}\wedge\mathbf{a}},\ \mathbf{C}\equiv\frac{\mathbf{a}\wedge\mathbf{b}}{\mathbf{c}\cdot\mathbf{a}\wedge\mathbf{b}}$$

so that if $\mathbf{r}=\alpha\mathbf{a}+\beta\mathbf{b}+\chi\mathbf{c}$, then $\alpha=\mathbf{A}\cdot\mathbf{r}$, $\beta=\mathbf{B}\cdot\mathbf{r}$, $\chi=\mathbf{C}\cdot\mathbf{r}$.



$$d_a = \frac{\mathbf{d}\cdot\mathbf{b}\wedge\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c}}$$

Easter term: These ideas are related to the linear algebra of matrices. If $\mathbf{M}$ is a matrix, then if $\mathbf{M}$ is orthogonal $\mathbf{M}^{-1} = \mathbf{M}^T$, then $\mathbf{M}^T\mathbf{M} = \mathbf{I}$. If $\mathbf{M}$ is not orthogonal and not singular, then $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$. In each case, the rows of $\mathbf{M}$ may be considered as the basis vectors, and the rows of $\mathbf{M}^T$ or $\mathbf{M}^{-1}$ are the *inverse basis*.

Cylindrical polar coordinates: (r,θ,z)

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$$

$$r \ge 0 \text{ and } 0 \le \theta < 2\pi \text{ (or } -\pi < \theta \le \pi\text{)}$$

Cylindrical polar coordinates: (r,θ,z) - right-handed, orthogonal

$$\mathbf{r} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Basis vectors

$$\begin{aligned}\hat{\mathbf{e}}_r &= \cos\theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}}\ ,\\ \hat{\mathbf{e}}_\theta &= -\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}}\ ,\\ \hat{\mathbf{e}}_z &= \hat{\mathbf{k}}\ .\end{aligned}$$

Self-isolation?

If you have been asked to self-isolate and do not have the lecture notes, please e-mail your Director of Studies (and cc me), telling them of your situation so we can get the notes to you.

Election of class representative?

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Self-nominate by Thursday 22 Oct

Make your 'campaign speech' on Saturday 25 Oct

Election Tuesday 27 Oct

Now Lecture 6: Vectors → Complex numbers

Recap of Lecture 6: Vectors → Complex numbers

Cylindrical polar coordinates

Basis vectors

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\,,$$
$$\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\,,$$
$$\hat{\mathbf{e}}_z = \hat{\mathbf{k}}\,.$$

Unit length: $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1$

Orthogonal: $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_r = 0$

The vector $\hat{\mathbf{e}}_\theta$ is the direction a point would move for a small increase in θ (for $r, z = const$). It has unit length (not units of radians).

$$\mathbf{r} = r\hat{\mathbf{e}}_\mathbf{r} + z\hat{\mathbf{e}}_\mathbf{z}, \quad |\mathbf{r}| = \sqrt{r^2 + z^2}$$

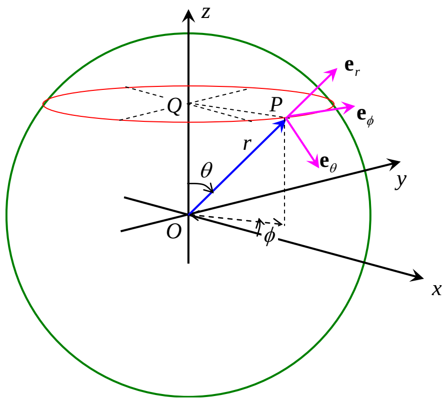
Note that $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ depend on θ and the basis changes depending on your location.

Important: Vectors have a direction and magnitude, but **not** a position. However, using the basis for polar coordinates requires a knowledge of your position and that you **change basis** if your position changes. This can rapidly lead to confusion! For vector algebra, it is far safer to compute things using a fixed **Cartesian basis**.

Plane polar (or circular polar) coordinates: (r,θ)

Plane polar coordinates: cylindrical polar without z.

Spherical polar coordinates: (r, θ, ϕ)



radius $r \geq 0$, $r = \sqrt{x^2 + y^2 + z^2}$

polar or inclination angle $0 \leq \theta \leq \pi$, $\theta = \cos^{-1} \dfrac{z}{r}$

azimuthal angle $0 \leq \phi < 2\pi$, $\phi = \tan^{-1} \dfrac{y}{x}$

Think about which quadrant for ϕ!

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r \cos\theta$$

$$\mathbf{r} = r \sin\theta \cos\phi \ \hat{\mathbf{i}} + r \sin\theta \sin\phi \ \hat{\mathbf{j}} + r \cos\theta \ \hat{\mathbf{k}}$$

$$\hat{\mathbf{e}}_r = \sin\theta(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}) + \cos\theta\hat{\mathbf{k}},$$
$$\hat{\mathbf{e}}_\theta = \cos\theta(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}) - \sin\theta\hat{\mathbf{k}},$$
$$\hat{\mathbf{e}}_\phi = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}.$$

$$\mathbf{r} = r\hat{\mathbf{e}}_{\mathbf{r}}$$

$\hat{\mathbf{e}}_r$ is a function of θ and ϕ, $\hat{\mathbf{e}}_\theta$ is a function of θ, and $\hat{\mathbf{e}}_\phi$ is a function of ϕ.

Example 21: Tripos question 2005, Paper 1, 5C

The vertices of a tetrahedron O, P, Q, R have coordinates $(0,0,0)$, $(2,1,1)$, $(1,2,2)$ and $(0,0,3)$ respectively. Find by vector methods

(a) the angle between the faces OPR and OQR [5]

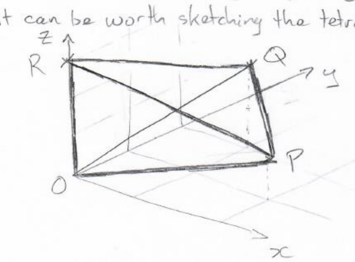
(b) the angle between the vector $\overrightarrow{OP}$ and the normal to the face PQR [5]

(c) the area of the face PQR [5]

(d) the shortest distance from the origin to the plane containing P, Q and R. [5]

Answer on Moodle (a) Cross product for normals
Dot product for angles
(b) Similar to (a)
(c) Area of a triangle is half that of a parallelogram
(d) $d = \overrightarrow{OP} \cdot \hat{n}$

It can be worth sketching the tetrahedron first



(a) OPR and OQR intersect on $\overrightarrow{OR}$, on the z-axis $\Rightarrow$ angle is the same as projection onto the $z=0$ plane of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, i.e. angle between $(2,1,0)$ and $(1,2,0)$

$$(2,1,0)\cdot(1,2,0) = 2+2 = 4 = \sqrt{5}\sqrt{5}\cos\theta$$
$$\Rightarrow \theta = \cos^{-1}\frac{4}{5}$$

We could have computed this using normals

$$\underset{\sim}{n}_{OPR} = \overrightarrow{OP}\times\overrightarrow{OR} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = (3,-6,0)$$

$$\underset{\sim}{n}_{OQR} = \overrightarrow{OQ}\times\overrightarrow{OR} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ 1 & 2 & 2 \\ 0 & 0 & 3 \end{vmatrix} = (6,-3,0)$$

$$\underset{\sim}{n}_{OPR}\cdot\underset{\sim}{n}_{OQR} = |\underset{\sim}{n}_{OPR}||\underset{\sim}{n}_{OQR}|\cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\frac{(3,-6,0)\cdot(6,-3,0)}{\sqrt{9+36}\,\sqrt{36+9}} = \cos^{-1}\frac{4}{5}$$

(b) Angle between $\overrightarrow{OP}$ and normal to PQR

$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = (1,2,2)-(2,1,1) = (-1,1,1)$$
$$\overrightarrow{PR} = \overrightarrow{OR}-\overrightarrow{OP} = (0,0,3)-(2,1,1) = (-2,-1,2)$$

$$\underset{\sim}{n}_{PQR} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ -1 & 1 & 1 \\ -2 & -1 & 2 \end{vmatrix}\begin{matrix} \\ -1 & 1 \\ -2 & -1 \end{matrix}$$

$$= \hat{\underset{\sim}{i}}(2+1) + \hat{\underset{\sim}{j}}(-2+2) + \hat{\underset{\sim}{k}}(1+2)$$
$$= (3,0,3)$$

$$\underset{\sim}{n}_{PQR}\cdot\overrightarrow{OP} = |\underset{\sim}{n}_{PQR}||\overrightarrow{OP}|\cos\phi$$

$$\Rightarrow \phi = \cos^{-1}\frac{(3,0,3)\cdot(2,1,1)}{\sqrt{18}\,\sqrt{6}} = \cos^{-1}\frac{9}{3\sqrt{12}}$$
$$= \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

(c) Area of PQR

$$A = \tfrac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \tfrac{1}{2}|\underset{\sim}{n}_{PQR}| = \tfrac{1}{2}|(3,0,3)|$$
$$= \frac{\sqrt{18}}{2} = \frac{3}{\sqrt{2}}$$

(d) Shortest distance to O of plane includin PQR

$$d = |\underset{\sim}{p}\cdot\hat{\underset{\sim}{n}}_{PQR}| \qquad \underset{\sim}{p} \text{ on plane}$$
$$= \overrightarrow{OP}\cdot\hat{\underset{\sim}{n}}_{PQR}$$
$$= \overrightarrow{OP}\cdot\frac{\underset{\sim}{n}_{PQR}}{|\underset{\sim}{n}_{PQR}|}$$
$$= \frac{(2,1,1)\cdot(3,0,3)}{\sqrt{18}}$$
$$= \frac{9}{3\sqrt{2}}$$
$$= \frac{3}{\sqrt{2}}$$

Recap of Lecture 5: Vectors $\rightarrow$ Complex numbers

2. Complex numbers

Domain of Integers: $\mathbb{Z}$ (from Zahlen, German for “numbers”)

Domain of Real numbers: $\mathbb{R}$

Domain of Complex numbers: $\mathbb{C}$

Domain of Imaginary numbers: $\mathbb{I}$

“Blackboard bold font”

See https://en.wikipedia.org/wiki/Blackboard_bold

Definition of i: $\sqrt{-1} \equiv i$,

The principal value of the square root:

Complex number: $z = x + iy$, $x, y \in \mathbb{R}$

Real part of $z = x + iy$: $x = \Re(z) = \mathrm{Re}(z) \in \mathbb{R}$

Imaginary part of $z = x + iy$: $y = \Im(z) = \mathrm{Im}(z) \in \mathbb{R}$

Addition and subtraction: similar to vectors

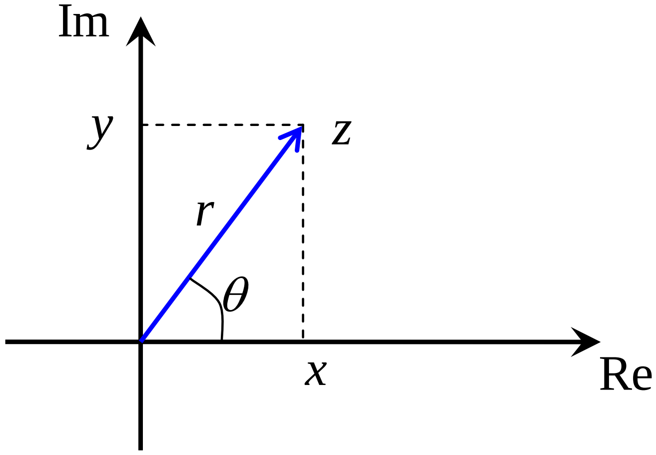
$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2), \quad \mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

If $c \in \mathbb{R}$, then $\mathrm{Re}(cz_1) = c\,\mathrm{Re}(z_1)$, $\mathrm{Im}(cz_1) = c\,\mathrm{Im}(z_1)$, but multiplication of complex numbers different from vectors

$$\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$$
$$\mathrm{Im}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Im}(z_1)\mathrm{Re}(z_2)$$

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$\mathbb{I} \times \mathbb{I} \to \mathbb{R}$$
$$\mathbb{R} \times \mathbb{I} \to \mathbb{I}$$

$$\mathbb{C} \times \mathbb{R} \to \mathbb{C}$$
$$\mathbb{C} \times \mathbb{I} \to \mathbb{C}$$
$$\mathbb{C} \times \mathbb{C} \to \mathbb{C}$$



Argand diagram:
$z = x + iy = r\cos\theta + ir\sin\theta$

Modulus of z: $|z| \equiv \mathrm{mod}(z) = \sqrt{x^2 + y^2} = r$

Argument of z: $\arg(z) = \tan^{-1}\dfrac{y}{x} = \theta$; think about quadrant for θ.

Principal argument generally $-\pi < \arg(z) \le \pi$

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Self-nominate by Thursday 22 Oct

Make your 'campaign speech' on Saturday 25 Oct

Election Tuesday 27 Oct

Now Lecture 7: Complex numbers

Recap of Lecture 7: Complex numbers

Complex conjugate: z^*

$$\mathrm{Re}(z^*) = \mathrm{Re}(z), \quad \mathrm{Im}(z^*) = -\mathrm{Im}(z)$$

$$|z^*| = |z|, \quad \arg(z^*) = -\arg(z)$$

$$\left(z^*\right)^* = z$$

Modulus: $zz^* = z^*z = |z|^2 = |z^*|^2$

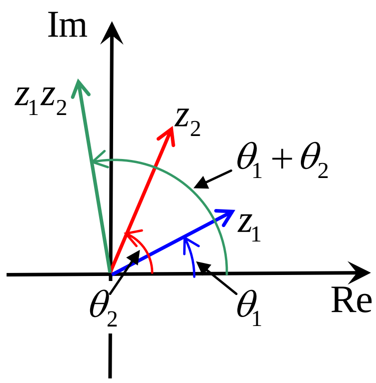
Complex exponential: $\cos\theta + i\sin\theta = e^{i\theta} = e^{i(\theta+2n\pi)}$, integer n

If $z = x + iy = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$ then

$$z^* = x - iy = r\left(\cos\theta - i\sin\theta\right) = r\left(\cos(-\theta) + i\sin(-\theta)\right) = re^{-i\theta}$$

Multiplication $z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Division $\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2}\dfrac{z_2^*}{z_2^*} = \dfrac{z_1 z_2^*}{|z_2|^2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$



If $z = re^{i\theta}$ and $a = se^{i\alpha}$, then $z^n = a$ can be written as ($n \in \mathbb{Z}$)

$$r^n e^{in\theta} = se^{i\alpha} = se^{i(\alpha+2m\pi)}$$

so $r = s^{1/n}$ and $\theta = \dfrac{\alpha + 2m\pi}{n}$ for $m = 0, 1, \cdots, n-1$.

De Moivre's theorem: $\cos\lambda\theta + i\sin\lambda\theta = (\cos\theta + i\sin\theta)^{\lambda}$ for $\lambda\in\mathbb{C}$.

Complex logarithms: $\ln(\exp(z)) \equiv \exp(\ln(z)) \equiv z \quad \forall z\in\mathbb{C}$

Since $z = |z|\exp(i\arg(z)) = |z|\exp(i[\arg(z)+2n\pi]), \quad n\in\mathbb{Z}$

then $\ln z = \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i(\theta+2n\pi), \quad n\in\mathbb{Z}$.

The 'principal value' is $\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta \quad (-\pi < \theta \le \pi)$

Geometric progression

$$S_N = \sum_{k=0}^{N-1} a\lambda^k = a\frac{1-\lambda^N}{1-\lambda} \qquad T_N = \sum_{k=1}^{N} a\lambda^k = a\lambda\frac{1-\lambda^N}{1-\lambda}$$

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Self-nominate by Thursday 22 Oct

- Sizhe Zhang (Churchill)

Make your 'campaign speech' on Saturday 25 Oct

Election Tuesday 27 Oct

Now Lecture 8: Complex numbers → Hyperbolic functions

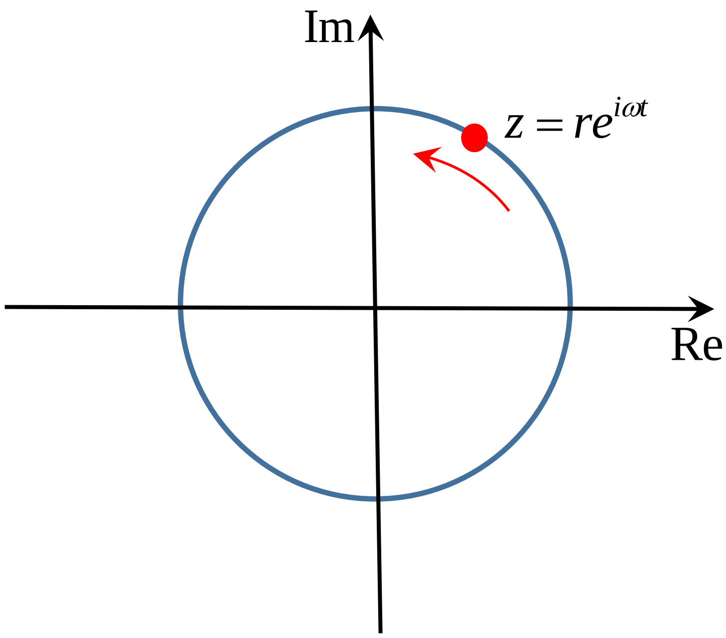
Lecture 8: Complex numbers → Hyperbolic functions

2. Complex numbers

Oscillations

$$
\begin{aligned}
x(t) &= \Re\left[Ae^{i\omega t}\right] = \Re\left[(a-ib)e^{i\omega t}\right] \\
&= \Re\left[(a-ib)(\cos\omega t + i\sin\omega t)\right] \\
&= \Re\left[a\cos\omega t + ia\sin\omega t - ib\cos\omega t - i^2 b\sin\omega t\right]. \\
&= \Re\left[a\cos\omega t + b\sin\omega t + i(a\sin\omega t - b\cos\omega t)\right] \\
&= a\cos\omega t + b\sin\omega t
\end{aligned}
$$

with $a,b \in \mathbb{R}$; $A = a - ib \in \mathbb{C}$.



Differentiation by a real variable works as normal so

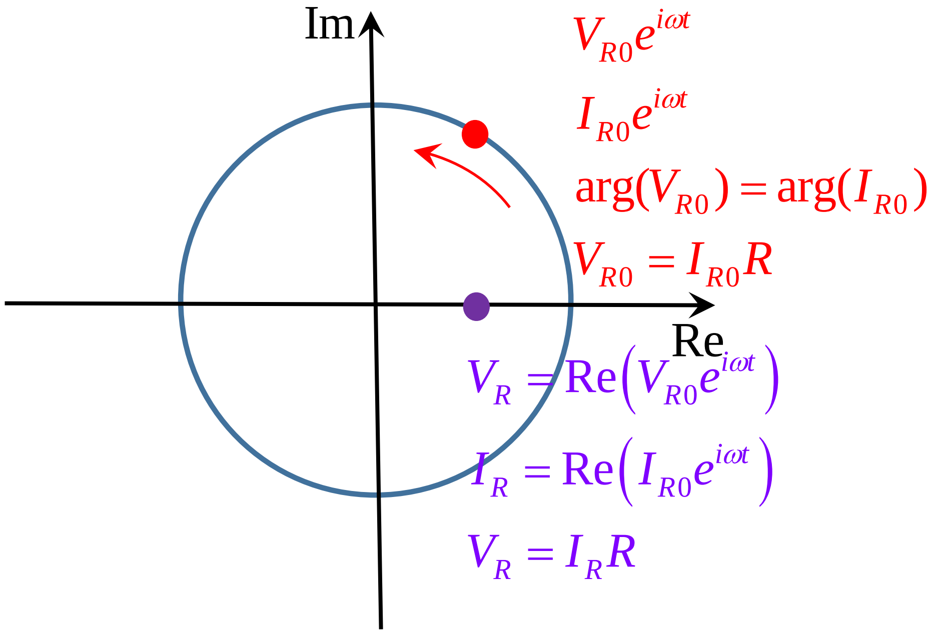
$$\frac{d}{dt}(\mathrm{Re}(z)) = \mathrm{Re}\left(\frac{dz}{dt}\right), \quad \frac{d}{dt}(\mathrm{Im}(z)) = \mathrm{Im}\left(\frac{dz}{dt}\right)$$

when $z \in \mathbb{C}$; $t \in \mathbb{R}$.

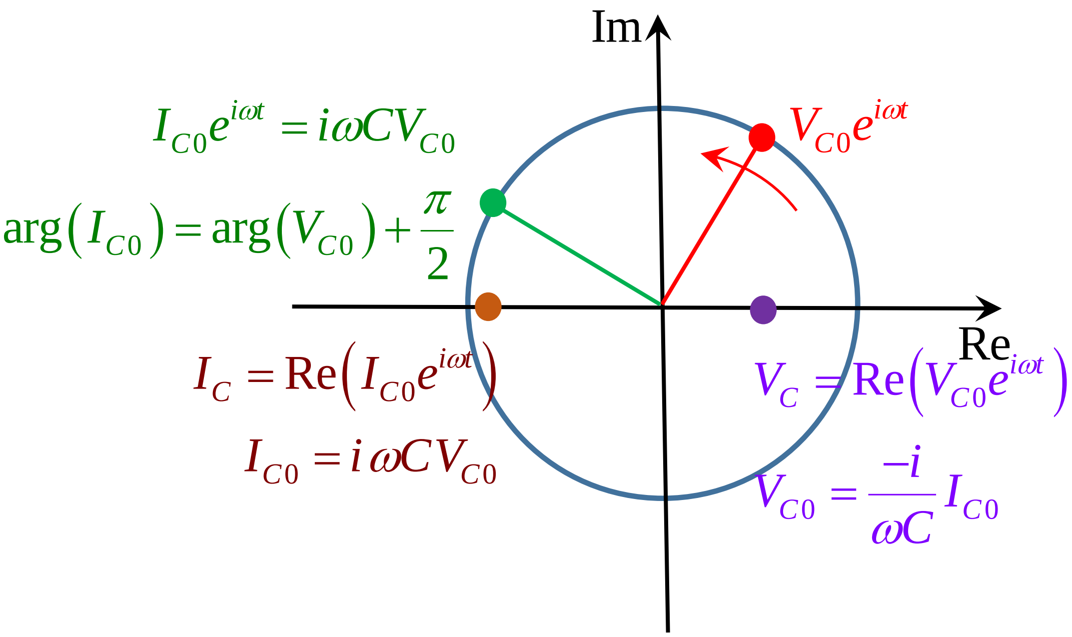
Example 32: Impedance of AC circuit

You do not need to know the electronics for this course!

The current through a resistor $R \in \mathbb{R}$ is given by Ohm's Law $I_R = V_R / R$ and so it is in phase with the voltage



The current through a capacity is $I_C = \dfrac{dQ}{dt} = C\dfrac{dV_C}{dt}$ so if $V_C = \mathrm{Re}\left(V_{C0}e^{i\omega t}\right)$ then $I_C = \mathrm{Re}\left(i\omega C V_{C0}e^{i\omega t}\right)$



Fundamental theorem of algebra:

If $P(z)$ is a polynomial of degree n,

$$P(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0,$$

then $P(z) = 0$ has n (complex) roots for all possible coefficients $a_0, a_1, a_2, \cdots, a_n \in \mathbb{C}$.

Equivalently, if $z = z_1$ is a root of $P(z) = 0$, then

$$P(z) = (z - z_1)Q(z) = 0$$

and $Q(z)$ is a polynomial of degree $n-1$, but also $Q(z) = 0$ must have at least one route, so $P(z) = (z - z_1)(z - z_2)R(z) = 0$, *etc.*

Note that roots may be repeated.

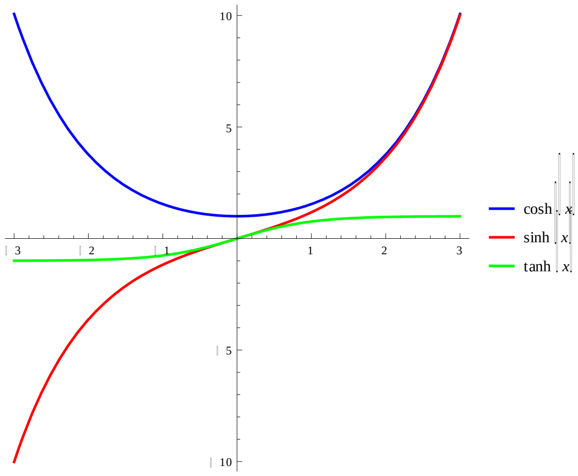
Lecture 8:* *Complex numbers* $\rightarrow$ *Hyperbolic functions

3. Hyperbolic functions

Hyperbolic cosine $\cosh x = \frac{1}{2}\left(\exp(x) + \exp(-x)\right)$

Hyperbolic sine $\sinh x = \frac{1}{2}\left(\exp(x) - \exp(-x)\right)$

Hyperbolic tangent $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}$



Relationship with normal trig functions:

$$\cos(iz) = \cosh z$$

$$\sin(iz) = i \sinh z$$

$$\tan(iz) = i \tanh z$$

Identities:

$$\cosh^2 x - \sinh^2 x \equiv 1$$

$$1 - \tanh^2 x \equiv \operatorname{sech}^2 x$$

$$\coth^2(x) - 1 \equiv \operatorname{cosech}^2(x)$$

$$\sinh^{-1} x \equiv \ln\left(x + \sqrt{x^2+1}\right), \quad \cosh^{-1} x \equiv \pm\ln\left(x + \sqrt{x^2-1}\right),$$

$$\tanh^{-1} x \equiv \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$$

For identities such as $\cosh(A+B)$, swap the sign compared with their normal circular trig equivalents where there is a product of two sine functions (but not where there is a single sine). For example,

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cosh(A+B) \equiv \cosh A \cosh B + \sinh A \sinh B$$

$$\sin(A+B) \equiv \sin A \cos B + \sin B \cos A$$
$$\sinh(A+B) \equiv \sinh A \cosh B + \sinh B \cosh A$$

Equations for a circle:

$x^2 + y^2 = r^2$; $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1$: centred on origin, radius r

$(x-x_0)^2 + (y-y_0)^2 = r^2$; $x = x_0 + r\cos\theta$, $y = y_0 + r\sin\theta$: centred on (x_0, y_0), radius r

$x^2 + ax + y^2 + by + c = 0$: centred on $\left(-\tfrac{1}{2}a, -\tfrac{1}{2}b\right)$, radius $r = \sqrt{\left(\tfrac{1}{2}a\right)^2 + \left(\tfrac{1}{2}b\right)^2 - c}$, *provided* $r \in \mathbb{R}$.

Equations for an ellipse:

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $x = a\cos\theta$, $y = b\sin\theta$: centred on origin. If $a > b$, then semi-major axis a and semi-minor axis b.

Now Lecture 9: Hyperbolic functions → Differentiation

Recap of Lecture 9: Hyperbolic functions → Differentiation

Equations for an hyperbola:

$$x^2 - y^2 = s^2;\ x = s\cosh\theta,\ y = s\sinh\theta,$$

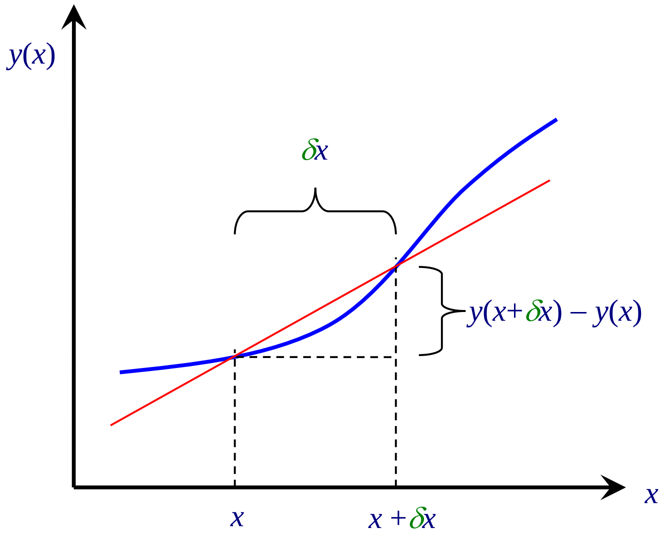
$$x^2 - y^2 = -s^2;\ x = s\sinh\theta,\ y = s\cosh\theta$$

For $ax^2 + bx + cy^2 + dy + e = 0$

- Circle if $a = c$ and (once completing the squares) have positive constant on right-hand side; no solution if negative constant.
- Ellipse if a, c have same sign (*i.e.* $ac > 0$) and (once completing the squares) have positive constant on right-hand side; no solution if negative constant.
- Hyperbola if a, c have opposite signs.

Recap of Lecture 9: Hyperbolic functions → Differentiation

Derivative of $y(x)$: $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{y(x+\delta x) - y(x)}{\delta x}$



For a function to be ***differentiable***, it must be continuous and

$$\lim_{h\to 0}\left(\frac{f(x)-f(x-h)}{h}\right)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Differential operator: $\dfrac{d}{dx}$

Product rule: $\dfrac{d}{dx}(fg)=\dfrac{df}{dx}g+f\dfrac{dg}{dx}$

Quotient rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right)=\dfrac{\dfrac{df}{dx}g-f\dfrac{dg}{dx}}{g^2}$

Leibnitz's formula:

$$\begin{aligned}\frac{d^n}{dx^n}(fg) &= f^{(n)}g^{(0)}+nf^{(n-1)}g^{(1)}+\frac{n(n-1)}{2!}f^{(n-2)}g^{(2)}+ \\ &\quad \cdots+nf^{(1)}g^{(n-1)}+f^{(0)}g^{(n)} \\ &= \sum_{m=0}^{n}\frac{n!}{(n-m)!m!}f^{(n-m)}g^{(m)}\end{aligned}$$

The following look like dealing with fractions (although subtler underneath)

Chain rule: $\dfrac{d}{dx}f(u(x))=\dfrac{df(u)}{du}\dfrac{du(x)}{dx}=\dfrac{df}{du}\dfrac{du}{dx}$

Reciprocal rule:

$$\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}$$

Implicit differentiation: If $g(y)=f(x)$, then

$$\frac{dg}{dx}=\frac{dg}{dy}\frac{dy}{dx}=\frac{df}{dx}\quad\Rightarrow\quad\frac{dy}{dx}=\frac{df}{dx}\bigg/\frac{dg}{dy}$$

Functions you should know how to differentiate:

$y = x^a$	$y' = ax^{a-1}$
$y = \exp(ax)$	$y' = a\exp(ax)$
$y = \ln(ax) = \ln a + \ln x$	$y' = \dfrac{1}{x}$

These are worth remembering, but can be derived from $y = \exp(ax)$

$y = \sin(ax)$	$y' = a\cos(ax)$
$y = \cos(ax)$	$y' = -a\sin ax$
$y = \sinh(ax)$	$y' = a\cosh(ax)$
$y = \cosh(ax)$	$y' = a\sinh ax$

Others worth remembering (but can be derived):

$y = \tan ax$	$y' = a\sec^2 ax$
$y = \tanh ax$	$y' = a\,\mathrm{sech}^2\, ax$

Typos in notes for earlier lectures

Lecture 5, p. 53:

$$\mathbf{S}_{OAD} = \tfrac{1}{2}\overset{\overrightarrow{OD}\times\overrightarrow{OA}}{\cancel{\overrightarrow{OA}\times\overrightarrow{OD}}} = \tfrac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cancel{0}\,\tfrac{1}{2} & \cancel{1}\,\tfrac{1}{2} & \tfrac{1}{2}\,\cancel{0} \\ \cancel{\tfrac{1}{2}}\,0 & \cancel{\tfrac{1}{2}}\,1 & 0\,\cancel{\tfrac{1}{2}} \end{vmatrix} = \frac{1}{2}\left[\hat{\mathbf{i}}\left(\tfrac{1}{2}-0\right)+\hat{\mathbf{j}}(0-0)+\hat{\mathbf{k}}\left(0-\tfrac{1}{2}\right)\right]$$

$$= \overset{\left(-\frac{1}{4},0,\frac{1}{4}\right)}{\cancel{\left(\tfrac{1}{4},0,-\tfrac{1}{4}\right)}},$$

as before. Note that we need to be careful in choosing the order $\overrightarrow{OD}\times\overrightarrow{OA}$ and not $\overrightarrow{OA}\times\overrightarrow{OD}$ to ensure we have the correct (outward) direction for the normal.

Lecture 8, p. 101:

This is an oscillation with amplitude $|A| = 2c$ and phase $\theta = -\pi/3$:

$$\cancel{x(t) = 2c\left(\cos\left(\omega t - \tfrac{1}{3}\pi\right)+\sin\left(\omega t - \tfrac{1}{3}\pi\right)\right).}$$

$$x(t) = 2c\cos\left(\omega t - \tfrac{1}{3}\pi\right)$$

Lecture 9, p. 128:

$$= \mp\frac{x-1}{\sqrt{9-(x-1)^2}}$$

If $x, y \in \mathbb{R}$,

only valid for $-2 < x < 4$

Gradient becomes infinite as we approach the limits.

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Now Lecture 10: Differentiation → Elementary analysis

Recap of Lecture 10: Differentiation → Elementary analysis

Leibnitz's formula:

$$\frac{d^n}{dx^n}(fg) = f^{(n)}g^{(0)} + nf^{(n-1)}g^{(1)} + \frac{n(n-1)}{2!}f^{(n-2)}g^{(2)} + \cdots + nf^{(1)}g^{(n-1)} + f^{(0)}g^{(n)}$$

$$= \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} f^{(n-m)}g^{(m)}$$

(Video of proof also available separately.)

Stationary point: where $dy/dx = 0$

Local minimum: $d^m y/dx^m = 0$, $1 \le m < n$ and $d^n y/dx^n > 0$, even n

Local maximum: $d^m y/dx^m = 0$, $1 \le m < n$ and $d^n y/dx^n < 0$, even n

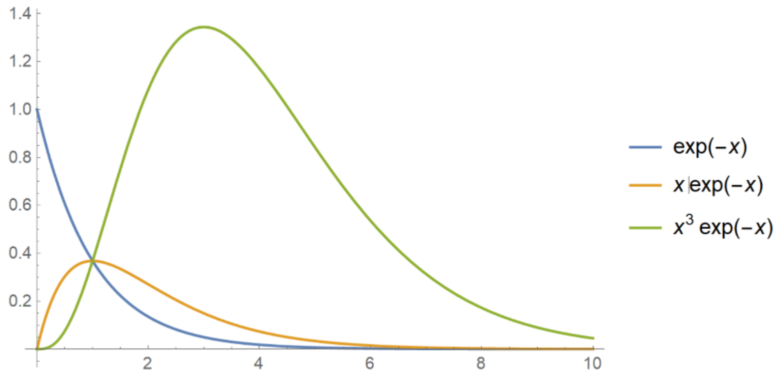
Stationary point of inflection: $d^m y/dx^m = 0$, $1 \le m < n$ and $d^n y/dx^n \ne 0$ for odd $n \ge 2$

Curve sketching

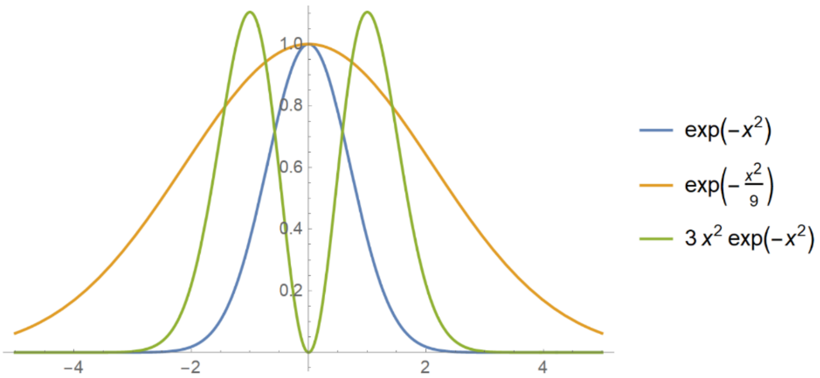
0. Over what range of x is the function defined?
1. Where are the intercepts with the x and y axes?
2. Are there any symmetries: does $y(-x) = y(x)$ (even) or $y(-x) = -y(x)$ (odd)?
3. What is the behaviour as $x \to \pm\infty$?
4. Are there any ***singularities*** (*i.e.* points where the function blows up to infinity)?
5. Where are the stationary points, and what is their nature?
6. Is the curvature positive or negative? Are there any points of inflection?

Often do not need to complete all these steps

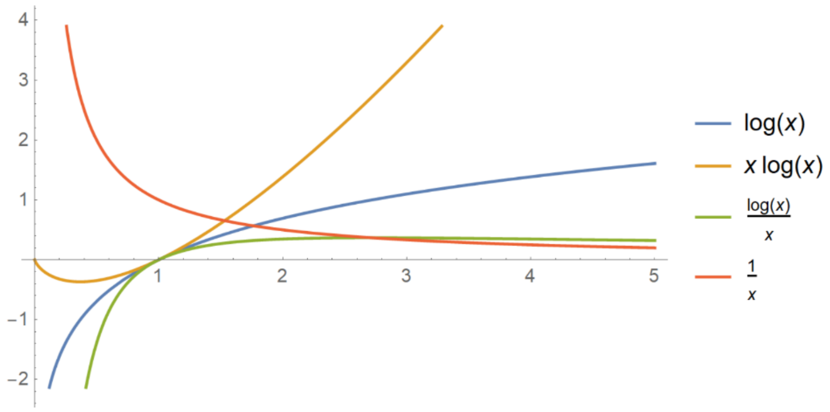
For $x \to \infty$, $x^\alpha e^{-x} \to 0$; $x^{-\alpha} e^x \to \infty$



Hence, for $x \to \infty$, $x^n e^{-x^\alpha} \to 0$

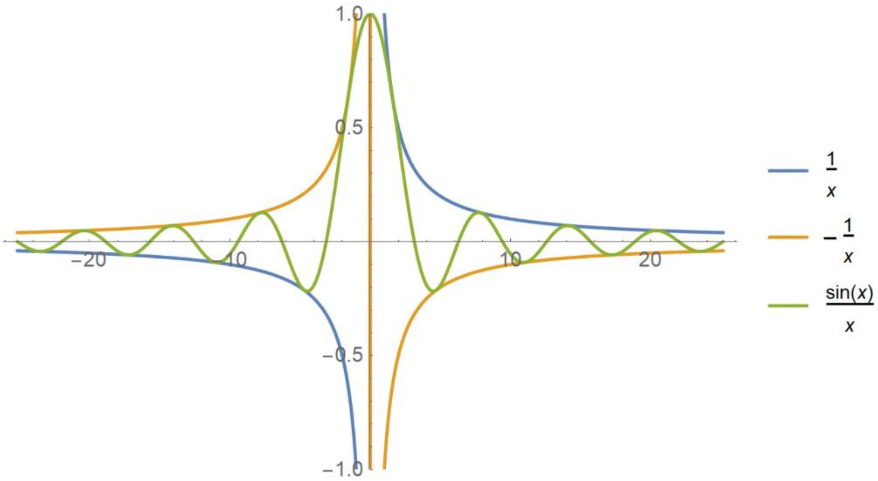


For $x \to \infty$, $\ln x / x^\alpha \to 0$



Recap of Lecture 10: Differentiation → Elementary analysis

If $y = \operatorname{sinc} x \equiv \frac{1}{x}\sin x$; does $\operatorname{sinc}(0) = 1$?

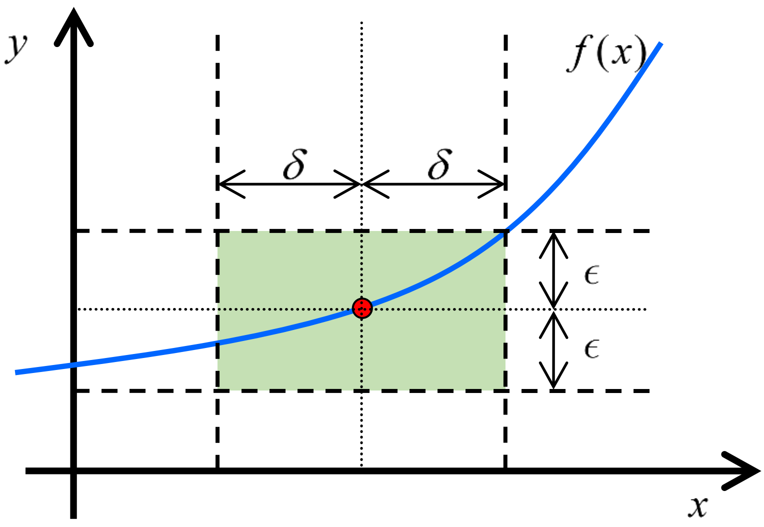


The limit is given by

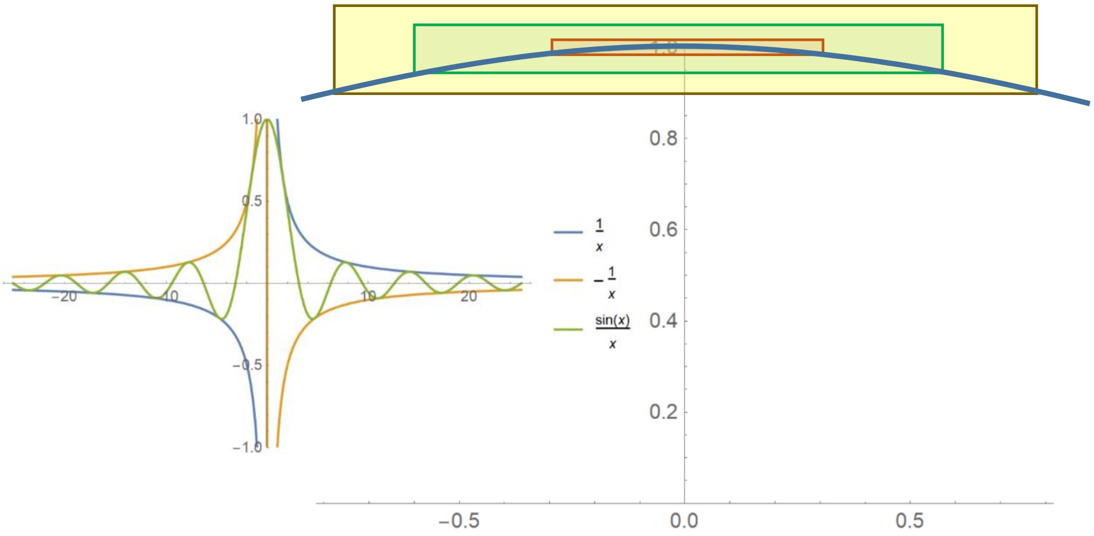
$$\lim_{x \to x_0} f(x) = K$$

if and only if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x) - K| < \epsilon \text{ for all } 0 < |x - x_0| < \delta. \qquad (108)$$



Limit can exist even if $f(x_0) \neq K$ or $f(x_0)$ is not defined.



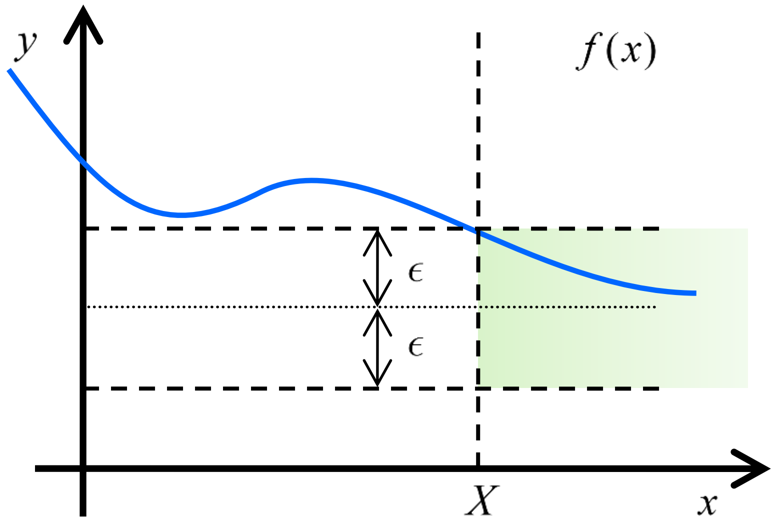
$\lim_{x\to 0} \operatorname{sinc}(x) = 1$, but still do not know if $\operatorname{sinc}(0) = 1$

The limit is given by

$$\lim_{x\to\infty} f(x) = K$$

if and only if for any $\epsilon > 0$, $\exists\, X < \infty$ such that

$$\left| f(x) - K \right| < \epsilon \text{ for all } x > X\,. \tag{109}$$



Wait, if not yet 9:00! (Apologies for Thursday)

Now Lecture 11: Elementary analysis → Infinite series

Recap of Lecture 11: Elementary analysis → Infinite series

Algebra of limits

Suppose $\lim_{x\to x_0} f(x) = F$ and $\lim_{x\to x_0} g(x) = G$, then

Addition/subtraction: $\lim_{x\to x_0}\left[f(x)\pm g(x)\right] = F \pm G$

Multiplication: $\lim_{x\to x_0}\left[f(x)g(x)\right] = FG$

Division:

$\lim_{x\to x_0}\left[f(x)/g(x)\right] = F/G$ if $G \neq 0$

$\lim_{x\to x_0}\left[f(x)/g(x)\right]$ does not exist if $F \neq 0$ and $G = 0$

$\lim_{x\to x_0}\left[f(x)/g(x)\right]$ may exist if $F = 0$ and $G = 0$

$\lim_{x\to x_0}\left[f(x)/g(x)\right]$ may exist if $F = \infty$ and $G = \infty$

Function: $\lim_{x\to x_0}\left[f\left(g(x)\right)\right] = f\left(\lim_{x\to x_0}\left[g(x)\right]\right) = f\left(G\right)$ iff $f\left(x\right)$ is continuous at $x = G$

Exponents: $\lim_{x\to x_0}\left[f(x)^{g(x)}\right] = \lim_{x\to x_0}\left[f(x)^{\lim_{x\to x_0} g(x)}\right] = F^G$

l'Hôpital's rule:

If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$

or $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = \infty$

then $\lim_{x\to x_0}\dfrac{f(x)}{g(x)} = \lim_{x\to x_0}\dfrac{f'(x)}{g'(x)}$; repeat if necessary

Big-O notation

At finite x:

For $f(x), g(x) \in \mathbb{R}$,

$$f(x) = O(g(x)) \text{ as } x \to a$$

if and only if $\exists$ constants $\epsilon, K > 0$ such that

$$|f(x)| \le K|g(x)| \text{ for all } |x-a| < \epsilon. \tag{110}$$

At infinity:

For $f(x), g(x) \in \mathbb{R}$,

$$f(x) = O(g(x)) \text{ as } x \to \infty$$

if and only if $\exists$ constants $X, K > 0$ such that

$$|f(x)| \le K|g(x)| \text{ for all } x > X. \tag{111}$$

If we say the $f(x) = O(x^2)$ when $x \to \infty$ then we say:
"*the function* $f(x)$ *is the order of* x^2 *as* x *approaches infinity*"

By convention, we take the tightest bound. So, if $f(x) = O(x^2)$ as $x \to \infty$, we would not say $f(x) = O(x^3)$, even though that satisfies (111).

$$\lim_{x\to 0}\left[a + bx + cx^2\right] = O(1) \text{ if } a \ne 0; \quad \lim_{x\to 0}\left[bx + cx^2\right] = O(x)$$

$$\lim_{x\to\infty}\left[a + bx + cx^2\right] = O(x^2);$$

$$\lim_{x\to\infty}[\cos x] = O(1); \quad \lim_{x\to\infty}[\sinh x] = O(\cosh x) = O(e^x)$$

Continuity

A real function $f(x)$ is continuous at $x = a$ if:

(i) $f(a)$ exists (*i.e.* the function is defined there), and
(ii) $\lim_{x \to a} f(x) = f(a)$,

i.e., the limit exists and is equal to the function. (112)

or

A real function $f(x)$ is continuous if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $|x - a| < \delta$. (113)

Limit (lecture 10)

The limit is given by

$$\lim_{x \to x_0} f(x) = K$$

if and only if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x) - K| < \epsilon \text{ for all } 0 < |x - x_0| < \delta. \qquad (108)$$

Recap of Lecture 11: Elementary analysis → Infinite series

6. Infinite series

Partial sum: $$S_n \equiv \sum_{k=0}^{n} u_k$$

If $\lim_{n\to\infty} S_n = S$, then for any ϵ, there exists a N such that
$$|S - S_n| < \epsilon \quad \forall\, n > N .$$

Series converges if $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} u_k = S$ is finite.

Need $u_k \to 0$ sufficiently rapidly as k increases

Series diverges if $\lim_{n\to\infty} S_n = \pm\infty$

Series may oscillate between a sequence of values as n increases

If $\sum_{k=0}^{\infty} u_k = S$ and $\sum_{k=0}^{\infty} v_k = T$ then $\sum_{k=0}^{\infty} (u_k + v_k) = S + T$.

$\sum_{k=0}^{\infty} (u_k + v_k) = R$ does not mean either $\sum_{k=0}^{\infty} u_k$ or $\sum_{k=0}^{\infty} v_k$ converge

Absolutely convergent if $\sum_{k=0}^{\infty} |u_k|$ converges. Changing the order of the terms has no effect.

Conditionally convergent if $\sum_{k=0}^{\infty} u_k$ converges but $\sum_{k=0}^{\infty} |u_k|$ does not. Changing the order of the terms may affect convergence.

If absolutely convergent then $\sum_{k=0}^{\infty} u_k$ is necessarily convergent.

Wait, if not yet 9:00! (Apologies for Thursday)

Now Lecture 12: Infinite series

Recap of Lecture 12: Infinite series

Grouping terms: does not change convergence; may help analysis.

Reordering terms: if series not absolutely convergent, then reordering may change whether the series converges.

Harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.

Alternating harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2$.

Comparison test

Compare the unknown positive series $\sum_k u_k$ $(u_k \geq 0)$ with a known positive series $\sum_k v_k$ $(v_k \geq 0)$. For constant $K \geq 0$:

(a) If $u_k \leq v_k \quad \forall\, k \geq K$ then $\sum_{k=0}^{\infty} u_k$ is convergent if $\sum_{k=0}^{\infty} v_k$ is convergent;

(b) If $u_k \geq v_k \quad \forall\, k > K$ and $\sum_{k=0}^{\infty} v_k$ diverges, then $\sum_{k=0}^{\infty} u_k$ also diverges. (116)

Ratio test

For a positive series $\sum_k u_k$:

$$\lim_{k\to\infty} u_{k+1}/u_k < 1, \quad \sum_k u_k \text{ converges}$$
$$\lim_{k\to\infty} u_{k+1}/u_k > 1, \quad \sum_k u_k \text{ diverges}$$
$$\lim_{k\to\infty} u_{k+1}/u_k = 1, \quad \sum_k u_k \text{ may converge}$$

The indeterminate (last) case requires a different test.

Leibnitz criterion

An alternating series $S = \sum_k (-1)^{k+1} a_k$ with $a_k > 0$ converges if a_k is monotonically decreasing for large enough k and $\lim_{k\to\infty} a_k = 0$.

Proof:

Consider a partial sum including an even number of terms:

$$S_{2n} = a_1 - a_2 + a_3 + \cdots - a_{2n}.$$

$$S_{2(n+1)} = a_1 - a_2 + \cdots + a_{2n} + a_{2n+1} - a_{2n+2}$$

Since $S_{2(n+1)} - S_{2n} = a_{2n+1} - a_{2n+2} > 0$, the S_{2n} are monotonically increasing. We can regroup the series

$$S_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{>0} - \underbrace{(a_4 - a_5)}_{>0} \cdots - a_{2n} < a_1$$

so the partial sums with an even number of terms are positive ($a_1 + a_3 + \cdots + a_{2n-1} > a_2 + a_4 + \cdots + a_{2n}$) bounded above (by a_1) and monotonically ~~decreasing~~, hence they have a finite limit $0 < S < a_1$ as $n \to \infty$. increasing

Power series

For $f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$, if the limit $L \equiv \lim_{k\to\infty} |a_{k+1}/a_k|$ exists, then:

(a) Series converges (absolutely) for $|x| < 1/L$;

(b) Series diverges for $|x| > 1/L$;

(c) The test is indeterminate for $|x| = 1/L$. (118)

Ratio test: $$\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = |x|\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = |x|L < 1$$

$$\Rightarrow \qquad |x| < \frac{1}{\lim_{k\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right|} = \frac{1}{L}$$

Beyond NST1A $f(z) = \sum_{k=0}^{\infty} a_k z^k;\ z \in \mathbb{C}$ has ***circle of convergence***

$$|z| < \frac{1}{\lim_{k\to\infty}|a_{k+1}/a_k|}$$

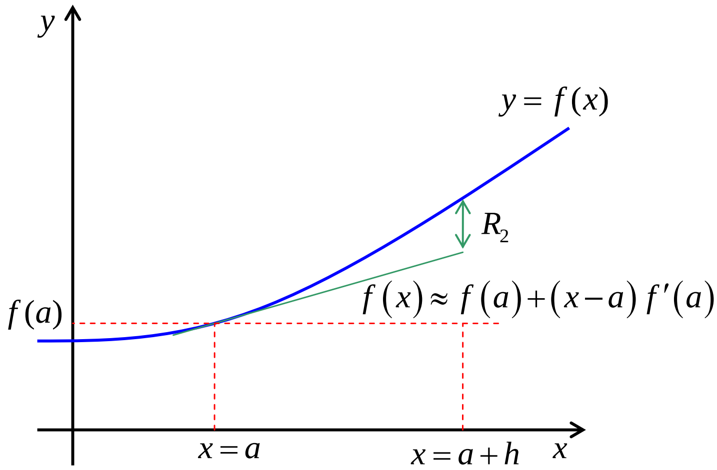
The third and final handout (Chapters 7 & 8) is available from the Centre for Mathematical Sciences

Wait, if not yet 9:00!

Now Lecture 13: Infinite series

Recap of Lecture 13: Infinite series

Taylor series



Taylor series

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{6} f'''(a)$$
$$+ \frac{(x-a)^4}{24} f^{(iv)}(a) + \frac{(x-a)^5}{120} f^{(v)}(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$
$$+ R_{n+1}\,.$$

Taylor theorem gives ***remainder term***

$$R_{n+1} = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\zeta)$$

for some $a \le \zeta \le x$ (or $x \le \zeta \le a$ if $x < a$),

$$\Rightarrow \qquad |R_{n+1}| \le \frac{|x-a|^{n+1}}{(n+1)!} \max_{a \le \zeta \le x} \left| f^{(n+1)}(\zeta) \right|.$$

Need to know how to use Taylor series and Taylor's theorem, but not how to derive Taylor's theorem.

Maclaurin series: Taylor series expanded about $x = 0$.

If $f(x)$ is infinitely differentiable, and remainder R_{n+1} goes to zero as $n \to \infty$, can uses an infinite number of terms to produce a ***Taylor power series***:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$+ \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

If $f(x)$ is **even** then the Power Series will only contain **even** powers of x.

If $f(x)$ is **odd** then the Power Series will only contain **odd** powers of x.

The derivative of an even function is an odd function.

The derivative of an odd function is an even function.

Common Taylor Series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.. + \frac{x^n}{n!} + \ldots. = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\exp(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots.. + \frac{(-1)^n x^n}{n!} + \ldots. = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!},$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots. + \frac{x^{2n}}{(2n)!} + \ldots. = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots. + \frac{x^{2n+1}}{(2n+1)!} + \ldots. = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Ratio test shows these series converge $\forall x$.

Absolute series: $\exp(x)$ for $x > 0$ and $\cosh x, \sinh x\ \forall x$.

Alternating series: $\exp(x)$ for $x < 0$ and $\cos x, \sin x\ \forall x$.

Reminder

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\cos x \sin \delta x + \cos(\delta x)\sin x - \sin(x)}{\delta x} \\
&= \cos x \underbrace{\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}}_{=1} = \cos x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos(x)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos(x)}{\delta x} \\
&= -\sin x \underbrace{\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}}_{=1} = -\sin x
\end{aligned}$$

The third and final handout (Chapters 7 & 8) is available from the Centre for Mathematical Sciences – **week days only**

Wait, if not yet 9:00!

Now Lecture 14: Infinite series

Recap of Lecture 14: Infinite series

De Moivre's theorem

$$\begin{aligned}\exp(\mathrm{i}\theta) &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] \\ &= \cos\theta + i\sin\theta\end{aligned}$$

Logarithms : cannot expand $\ln x$ about $x = 0$. Instead

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots, \quad -1 < x \le 1$$

For large x, use $\ln x = -\ln\frac{1}{x} = -\ln\left(1 - \frac{x-1}{x}\right) = -\ln(1-\xi)$ as

$\xi = \frac{x-1}{x} < 1$ when $x > 1 \rightarrow \ln x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots$

Binomial expansion for $f(x) = (1+x)^\alpha$, for real α

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \\ &\quad + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\ldots(\alpha-n+1)}{n!}x^n + \cdots.\end{aligned}$$

If $\alpha \in \mathbb{N}$, then valid for any x, but if $\alpha \notin \mathbb{N}$, then valid only for $-1 < x \le 1$.

Compose more complex series from combinations of simpler ones:

$$\mathit{series}\left[\frac{\log(1+x)}{1-x},x\right]=\mathit{series}\left[\log(1+x),x\right]\times\mathit{series}\left[\frac{1}{1-x},x\right]$$

$$\mathit{series}\left[\frac{1}{1+\sin x},x\right]=\mathit{series}\left[\frac{1}{1+\xi},\xi\right] \text{ where } \xi=\mathit{series}[\sin x,x]$$

Composing can be much simpler than determining derivatives

$$\mathit{series}[\log(\cos x),x]=\log[\cos(0)]-x\tan(0)-\frac{x^2}{2!}\sec^2(0)-\frac{x^3}{3!}\frac{\sin(0)}{\cos^3(0)}+\cdots$$

$$\begin{aligned}\mathit{series}[\log(\cos x),x]&=\mathit{series}\left[\log(1+(\cos x-1)),x\right]\\&=\mathit{series}\left[\log(1+\xi),\xi\right],\quad \xi=\mathit{series}[\cos x-1,x]\end{aligned}$$

Can differentiate and integrate series

$$\mathit{series}[df/dx,x]=\frac{d}{dx}\mathit{series}[f(x),x]$$

$$\mathit{series}\left[\int f\,dx,x\right]=\int \mathit{series}[f(x),x]dx$$

$$f(x)=(1+x)^{-1}=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$$

$$\int f(x)dx=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots+(-1)^n\tfrac{1}{n+1}x^{n+1}+\cdots=\ln(1+x)$$

Example 62: Power series

Find the power series expansion about $x=0$ of $(2+x)^{-1/2}$.

Write in standard form

$$(2+x)^{-1/2} = \frac{1}{\sqrt{2}}\left(1+\tfrac{1}{2}x\right)^{-1/2} = \frac{1}{\sqrt{2}}(1+y)^{\alpha}$$

with $y=\frac{1}{2}x$ and $\alpha=-\frac{1}{2}$.

Binomial expansion therefore

$$(2+x)^{-1/2} = \frac{1}{\sqrt{2}}\left[1+\alpha y+\frac{\alpha(\alpha-1)}{2!}y^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}y^3+\cdots\right.$$
$$\left.+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)\ldots(\alpha-n+1)}{n!}y^n+\cdots\right]$$

$\alpha = -\frac{1}{2}$
$y = \frac{1}{2}x$

$$(2+x)^{-1/2} = \frac{1}{\sqrt{2}}\left[1-\frac{1}{2}\left(\frac{x}{2}\right)+\frac{-\frac{1}{2}(-\frac{3}{2})}{2!}\left(\frac{x}{2}\right)^2+\frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{x}{2}\right)^3+\cdots\right.$$
$$\left.+\frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})\ldots(-\frac{1}{2}(1+2n-2))}{n!}\left(\frac{x}{2}\right)^n+\cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1-\frac{x}{4}+\frac{3x^2}{32}-\frac{5x^3}{128}+\cdots\right]$$

Converges for $-1 < y \le 1$
$\Rightarrow -2 < x \le 2$

This is an ~~alternating series~~ for $x>0$ ~~with monotonically decreasing terms with $a_k \to 0$ as $k \to \infty$ and so will converge. $\forall x$~~

~~$x<0$, need to consider ratio test.~~

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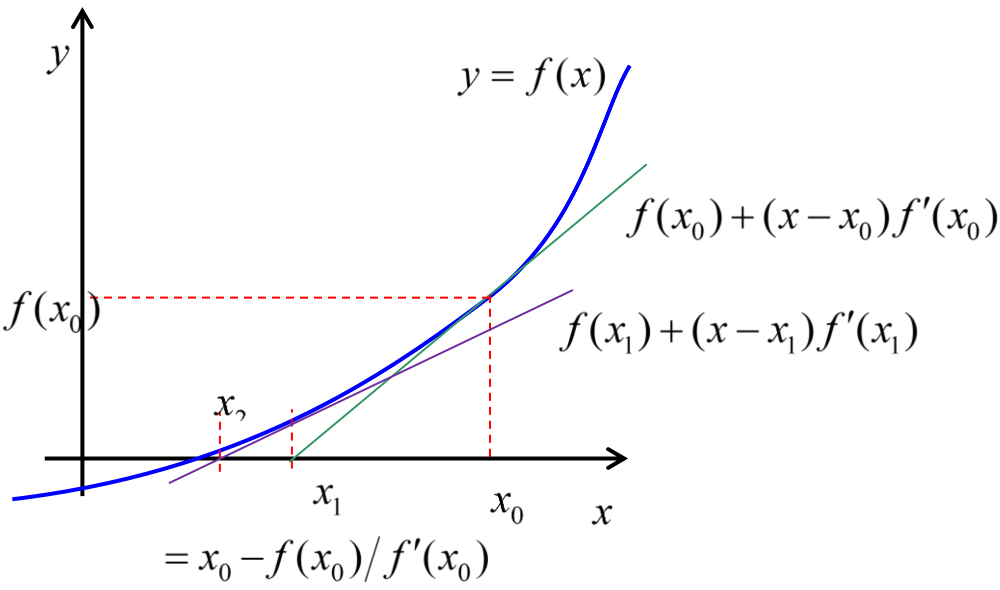
Wait, if not yet 9:00!

Now Lecture 15: Infinite series → Integration

Recap of Lecture 15: Infinite series $\to$ Integration

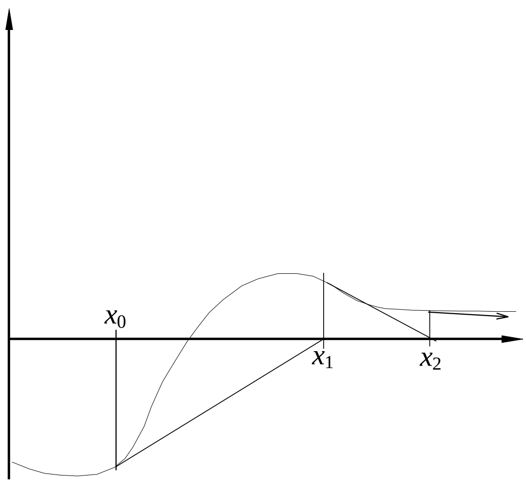
Newton-Raphson



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

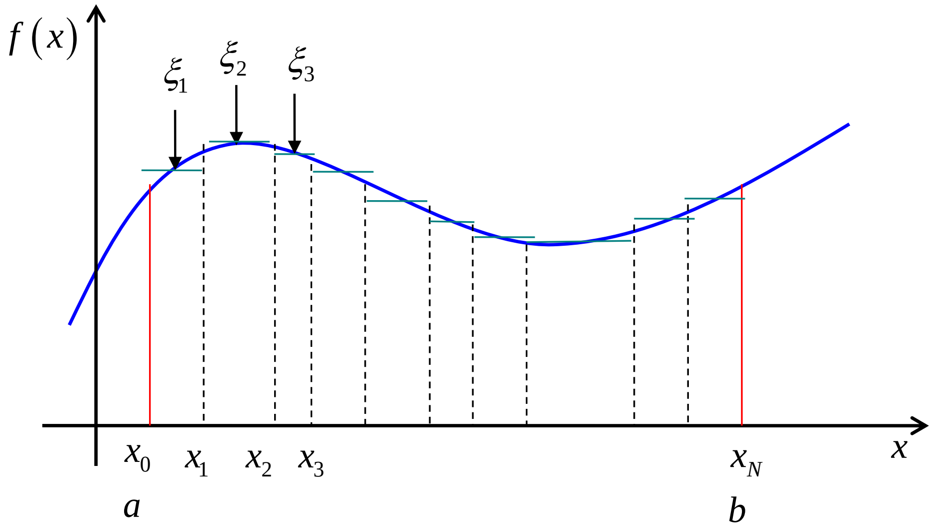
Quadratic convergence $\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n^2 \dfrac{f''(x_*)}{f'(x_*)} + O(\varepsilon_n^3)$ near root $f(x_*) = 0$, where $\varepsilon_n = x_n - x_*$.

Stationary points can cause a problem



Recap of Lecture 15: Infinite series → Integration

5. Integration



Riemann sum: $S_N = \sum_{i=1}^{N} (x_i - x_{i-1})\, f(\xi_i)$

Riemann integral: $\int_a^b f(x)\,dx \equiv \lim_{N\to\infty} \left[\sum_{i=1}^{N} (x_i - x_{i-1})\, f(\xi_i) \right] \equiv \lim_{N\to\infty} S_N$

If $f(x)$ and $g(x)$ are (Riemann) integrable in $[a,b]$ then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx\text{, where } k \text{ is constant}$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Integration is a linear operation (as is differentiation).

Integrand $f(x)$ is the function being integrated.

Primitive $F(x)$ is the integral of the integrand: $F(x) = \int_a^x f(u)\,du$.

Fundamental theorem of calculus: $\dfrac{dF(x)}{dx} \equiv \dfrac{d}{dx}\int_a^x f(u)\,du = f(x)$

If $F(x)$ is a primitive of $f(x)$, then $G(x) = F(x) + c$ is also a primitive: $\dfrac{d}{dx}F(x) = \dfrac{d}{dx}G(x) = f(x)$

Infinite number of primitives, differing by an additive constant.

If the function $f(x)$ is integrable for all $x \geq a$ then the **infinite integral** is

$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty}\left[\int_a^b f(x)\,dx\right],$$

provided the limit exists.

If $f(x)$ is singular at $x = x_*$ with $a \leq x_* \leq b$, then the **improper integral** is defined as

$$\int_a^b f(x)\,dx \equiv \lim_{\epsilon\to 0}\left[\int_a^{x_*-\epsilon} f(x)\,dx + \int_{x_*+\epsilon}^b f(x)\,dx\right]$$

provided the limits exist.

If $f(x)$ is **discontinuous** at $x = x_*$ with $a < x_* < b$, then the integral is

$$\int_a^b f(x)\,dx \equiv \int_a^{x_*} f(x)\,dx + \int_{x_*}^b f(x)\,dx.$$

Common integrals

$$\frac{dx^n}{dx} = nx^{n-1} \quad \Rightarrow \quad \int x^m\,dx = \frac{x^{m+1}}{m+1} + c,\ m \neq -1$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \ln|x| + c$$

$$\frac{d}{dx}\exp(ax) = a\exp(ax) \quad \Rightarrow \quad \int \exp(bx)\,dx = \frac{1}{b}\exp(bx) + c$$

$$\frac{d}{dx}\sin(ax) = a\cos(ax) \quad \Rightarrow \quad \int \cos(bx)\,dx = \frac{1}{b}\sin(bx) + c$$

$$\frac{d}{dx}\cos(ax) = -a\sin(ax) \quad \Rightarrow \quad \int \sin(bx)\,dx = -\frac{1}{b}\cos(bx) + c$$

$$\frac{d}{dx}\tan(ax) = a\sec^2(ax) \quad \Rightarrow \quad \int \sec^2(bx)\,dx = \frac{1}{b}\tan(bx) + c$$

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax) \quad \Rightarrow \quad \int \cosh(bx)\,dx = \frac{1}{b}\mathrm{sinhh}(bx) + c$$

$$\frac{d}{dx}\cosh(ax) = a\sinh(ax) \quad \Rightarrow \quad \int \sinh(bx)\,dx = \frac{1}{b}\cosh(bx) + c$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax) \quad \Rightarrow \quad \int \mathrm{sech}^2(bx)\,dx = \frac{1}{b}\tanh(bx) + c$$

Often use substitutions rather than simply remembering

$$\frac{d}{dx}\left(\sin^{-1}(ax)\right) = \frac{a}{\sqrt{1-a^2x^2}} \qquad \int \frac{dx}{\sqrt{b^2-x^2}} = \sin^{-1}\frac{x}{b} + c$$

$$[x = b\sin\theta]$$

$$\frac{d}{dx}\left(\cos^{-1}(ax)\right) = \frac{-a}{\sqrt{1-a^2x^2}} \Rightarrow \int \frac{dx}{\sqrt{b^2-x^2}} = -\cos^{-1}\frac{x}{b} + \hat{c}$$

$$[x = b\cos\theta]$$

$$\frac{d}{dx}\left(\tan^{-1}(ax)\right) = \frac{a}{1+a^2x^2} \Rightarrow \int \frac{dx}{b^2+x^2} = \frac{1}{b}\tan^{-1}\frac{x}{b} + \hat{c}$$

$$[x = b\tan\theta]$$

$$\frac{d}{dx}\left(\sinh^{-1}(ax)\right) = \frac{a}{\sqrt{a^2x^2+1}} \Rightarrow \int \frac{dx}{\sqrt{x^2+b^2}} = \sinh^{-1}\frac{x}{b} + c$$

$$[x = b\sinh\theta]$$

$$\frac{d}{dx}\left(\cosh^{-1}(ax)\right) = \frac{a}{\sqrt{a^2x^2-1}} \Rightarrow \int \frac{dx}{\sqrt{x^2-b^2}} = \cosh^{-1}\frac{x}{b} + c$$

$$[x = b\cosh\theta]$$

Special forms

$$\frac{d}{dx}\left[f(x)^{\alpha} \right] = \alpha \frac{df}{dx} f(x)^{\alpha-1} \Rightarrow \int \frac{df}{dx}\left[f(x) \right]^{\beta} dx = \frac{1}{\beta+1}\left[f(x) \right]^{\beta+1} + c$$

$$\beta \neq -1$$

$$\frac{d}{dx}\ln\left[f(x) \right] = \frac{f'(x)}{f(x)} \quad \Rightarrow \quad \int \frac{f'(x)}{f(x)} dx = \ln\left| f(x) \right| + c$$

For $\int (\cos x)^n\,dx$, $\int (\sin x)^n\,dx$, $\int (\cosh x)^n\,dx$, $\int (\sinh x)^n\,dx$, *etc.*, convert using trig ids to terms involving simpler powers (*e.g.* $\cos px$, $\sin px$, $\cosh px$, $\sinh px$, *etc.* with $0 \le p \le n$, $p, n \in \mathbb{Z}$) and/or $f'(x) f(x)^q$ then integrate, ***or*** use (complex) exponentials/De Moivre's theorem.

Wait, if not yet 9:00!

Now Lecture 16: Integration

Recap of Lecture 16: Integration

There is *always* more than one answer to an indefinite integral as things can differ by an arbitrary constant. Sometimes, they can look really quite different (especially if trig functions are involved), depending on your integration strategy:

Partial fractions

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x(x+1)}\,dx = \int \frac{\alpha}{x} + \frac{\beta}{x+1}\,dx = \alpha \ln|x| + \beta \ln|x+1| + c$$

either $\dfrac{\alpha}{x} + \dfrac{\beta}{x+1} = \dfrac{\alpha(x+1)+\beta(x)}{x(x+1)} = \dfrac{x(\alpha+\beta)+\alpha}{x(x+1)} \Rightarrow \alpha = 1,\ \beta = -1$

or 'cover-up rule' if can fully factorise denominator

$$x = 0 \quad \Rightarrow \quad \alpha = \frac{1}{0+1} = 1; \qquad x = -1 \quad \Rightarrow \quad \beta = \frac{1}{-1} = -1.$$

Cover-up rule more difficult if repeated root and not recommended.

Substitution

For $\int f(x)\,dx$, choose substitution

$$x = g(u) \Rightarrow dx = g'(u)\,du$$

and rewrite $f(x)$ in terms of u, *i.e.* $f(x) = f(g(u))$, so

$$\int f(x)\,dx = \int f(g(u))\frac{dg(u)}{du}\,du$$

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\frac{dg(u)}{du}\,du$$

Denominator involves	Substitution	Comments
$a^2 + x^2$	$x = a\tan\theta$	
$\sqrt{(a^2 - x^2)}$	$x = a\sin\theta$	Need $\lvert x\rvert < \lvert a\rvert$
$\sqrt{(x^2 - a^2)}$	$x = a\cosh\theta$	Need $\lvert x\rvert > \lvert a\rvert$
$\sqrt{(a^2 + x^2)}$	$x = a\sinh u$	
$a^2 - x^2$	$x = a\tanh u$	Need $\lvert x\rvert < \lvert a\rvert$
$a^2 - x^2$	$x = a\coth u$	Need $\lvert x\rvert > \lvert a\rvert$
$a^2 - x^2 = (a+x)(a-x)$	Partial fractions	
$a + bx + cx^2$ or $\sqrt{a + bx + cx^2}$	Complete the square	

For integrals of trig functions

$$\tan(x/2) = t\,,$$

$$\rightarrow \qquad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}\,.$$

Complex exponentials

Mix of trig and exponential functions can be easiest using complex exponentials (rather than integrating by parts repeatedly)

$$\int e^{ax}\sin bx\,dx = \mathrm{Im}\left[\int e^{(a+ib)x}\,dx\right]$$

Integration by parts

$$\int_a^b f\,\frac{dg}{dx}\,dx = \left[\,fg\,\right]_a^b - \int_a^b \frac{df}{dx}\,g\,dx$$

One route to Taylor series

Reduction formulae

Aim for a recurrence relation, often integrating by parts

$$I_n \equiv \int_a^b f(x;n)g'(x)\,dx = \left[\,f(x;n)g(x)\right]_a^b - \underbrace{\int_a^b f'(x;n)g(x)\,dx}_{\text{Rewrite in terms of } I_n, I_{n-1}, I_{n-2}\ldots}$$

$\rightarrow \qquad I_n = \Phi(I_{n-1}, I_{n-2}, \ldots)$ and evaluate $I_0, I_1, \ldots$

Often useful for integrands of the form $f(x;n) = \left[p(x)\right]^n q(x)$, *e.g.*

$$\int x^n \ln x\,dx$$

Double factorials

(You only need to know ordinary factorials)

$$m! = 1\times 2\times 3\times\cdots\times m$$

$$n!! = \begin{cases} 2\times 4\times 6\times\cdots\times n = 2^m(m!) & n = 2m \\ 1\times 3\times 5\times\cdots\times n = \dfrac{(2m+1)!}{2^m(m!)} & n = 2m+1 \end{cases}$$

Odd and even functions

Odd: $\displaystyle\int_{x_0-a}^{x_0+a} f(x) = 0 \text{ if } f(x-x_0) = -f(x_0-x)$

Even: $\displaystyle\int_{x_0-a_0}^{x_0+a} g(x)\,dx = 2\int_{x_0}^{x_0+a} g(x)\,dx \text{ if } g(x-x_0) = g(x_0-x)$

More partial fractions

Can do partial fractions using complex domain for factorisation of denominator (***not recommended: too easy to get it wrong!***):

$$\frac{x+1}{(1-x)(1+x^2)} = \frac{x+1}{(1-x)(x+i)(x-i)} = \frac{a}{1-x} + \frac{b}{x+i} + \frac{c}{x-i}$$

$$x = 1 \quad \Rightarrow \quad a = \frac{1+1}{(1+i)(1-i)} = \frac{2}{2} = 1$$

$$x = -i \quad \Rightarrow \quad b = \frac{-i+1}{(1+i)(-i-i)} = \frac{-i+1}{-2i(1+i)}\frac{i}{i} = \frac{1+i}{2(1+i)} = \frac{1}{2}$$

$$x = i \quad \Rightarrow \quad c = \frac{i+1}{(1-i)2i} = \frac{i+1}{2(i+1)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{x+1}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{1/2}{x+i} + \frac{1/2}{x-i}$$

To recover real domain answer, combine last two terms (although you could choose to integrate the complex partial fractions first!):

$$\frac{x+1}{(1-x)\left(1+x^2\right)}=\frac{1}{1-x}+\frac{\frac{1}{2}(x-i)+\frac{1}{2}(x+i)}{(x+i)(x-i)}=\frac{1}{1-x}+\frac{x}{x^2+1}$$

Easier to use the naïve way!

Integrating complex logarithm

If $z \in \mathbb{C}$ then $\int \frac{1}{z} dz = \ln z + c$ with $c \in \mathbb{C}$.

$$\begin{aligned}\int_{-3}^{-2} \frac{1}{z} dz &= [\ln z]_{-3}^{-2} = [\ln z]_{3e^{i\pi}}^{2e^{i\pi}} \\ &= \ln(2e^{i\pi}) - \ln(3e^{i\pi}) \\ &= \left(\ln\left|2e^{i\pi}\right| + i\pi\right) - \left(\ln\left|3e^{i\pi}\right| + i\pi\right) \\ &= \ln 2 - \ln 3 + i(\pi - \pi) \\ &= \ln|-2| - \ln|-3|\end{aligned}$$

Integration in the complex plane is not part of NST1A

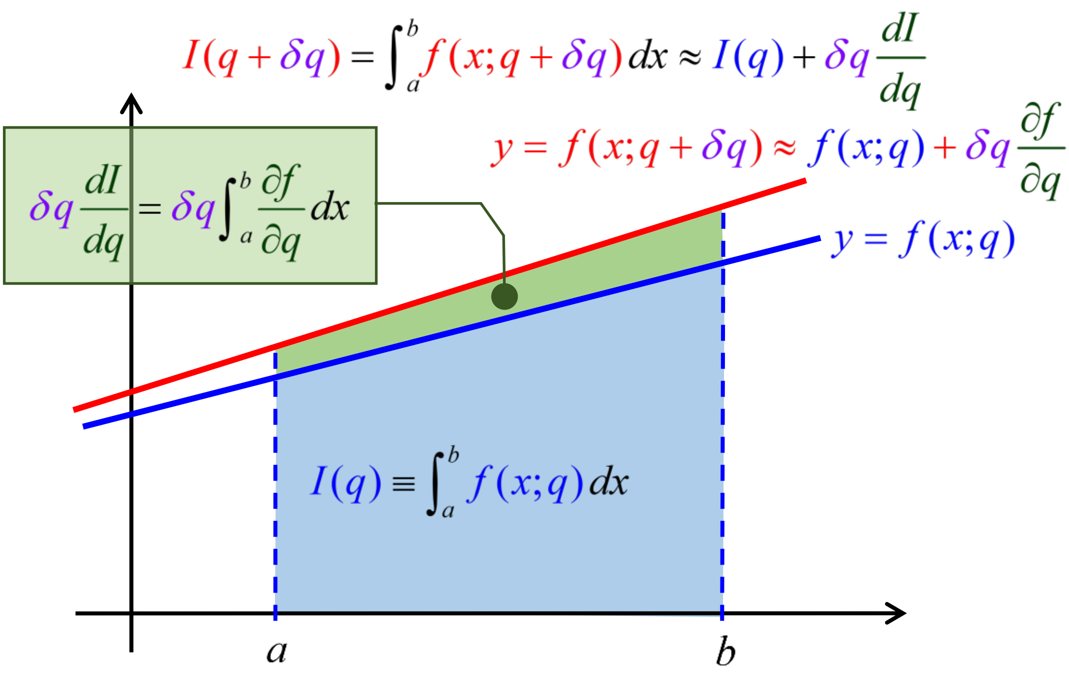
Now Lecture 17: Integration

Recap of Lecture 17: Integration

Differentiation by parameter

Three contributions: integrand and the two limits

Integrand depending on parameter $f(x;q)$



Note $\dfrac{\partial f}{\partial q}$ will generally be a function of x.

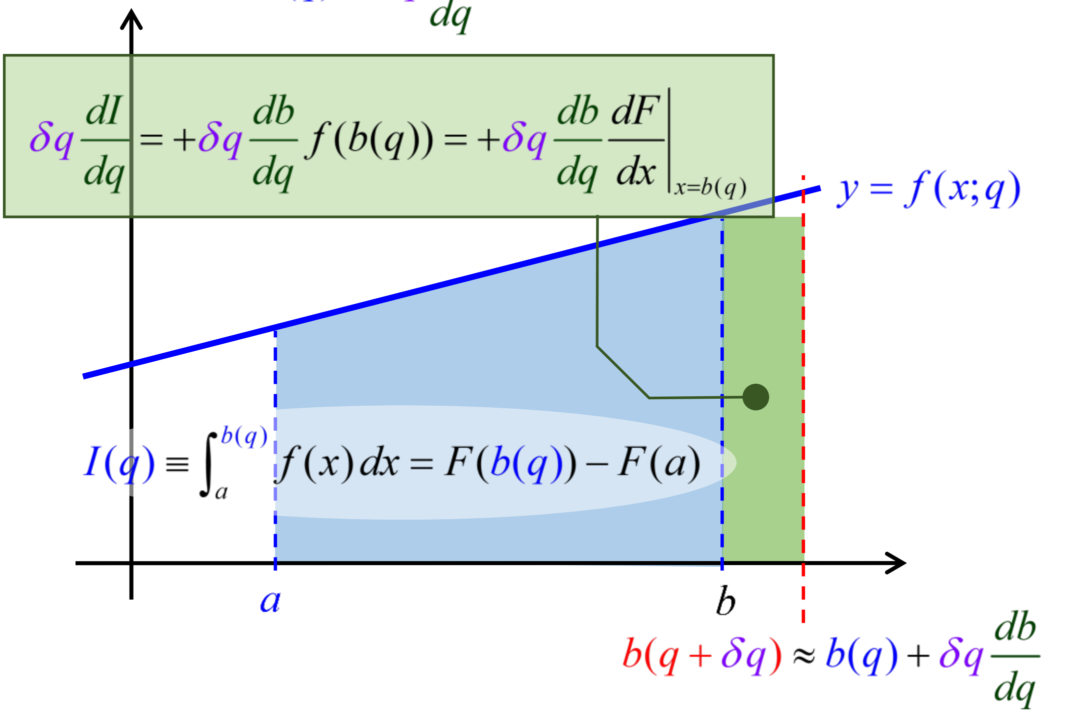
Upper limit depending on parameter $b(q)$

$$I(q+\delta q) = \int_a^{b(q+\delta q)} f(x)\,dx = F(b(q+\delta q)) - F(a)$$

$$\approx \left(F(b(q)) + \delta q \frac{db}{dq}\frac{dF}{dx}\bigg|_{x=b(q)} \right) - F(a)$$

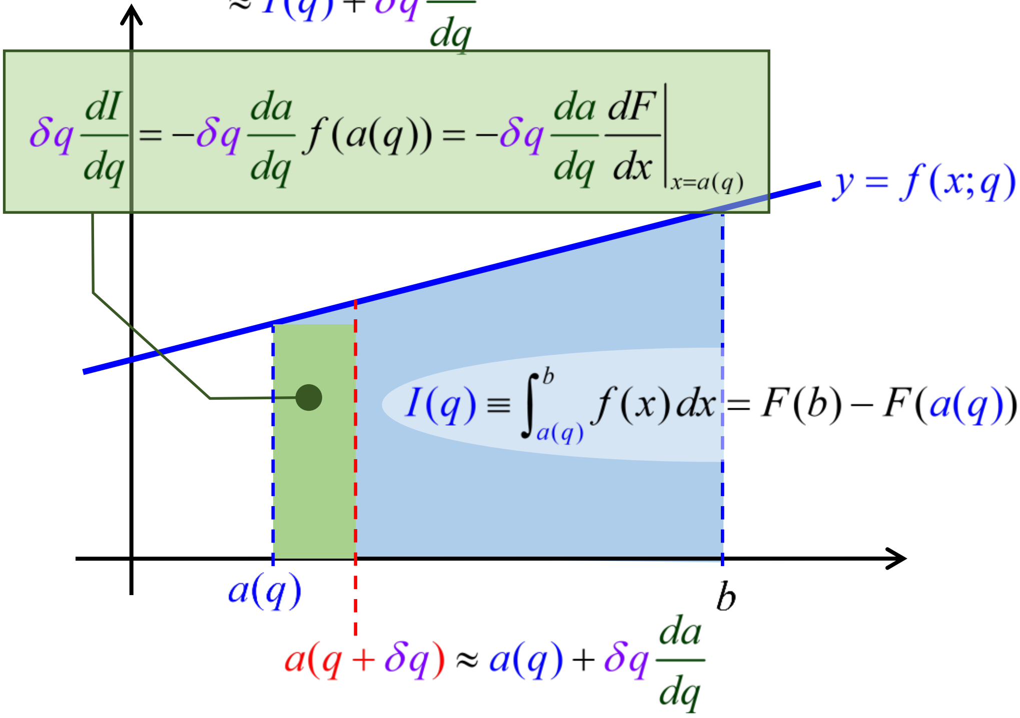
$$\approx \int_a^{b(q)+\delta q \frac{da}{dq}} f(x)\,dx = \int_a^{b(q)} f(x)\,dx + \int_{b(q)}^{b(q)+\delta q \frac{db}{dq}} f(x)\,dx$$

$$\approx I(q) + \delta q \frac{dI}{dq}$$



Lower limit depending on parameter $a(q)$

$$
\begin{aligned}
I(q+\delta q) &= \int_{a(q+\delta q)}^{b} f(x)\,dx = F(b) - F(a(q+\delta q)) \\
&\approx F(b) - \left(F(a(q)) + \delta q \frac{da}{dq}\frac{dF}{dx}\bigg|_{x=a(q)} \right) \\
&\approx \int_{a(q)+\delta q\frac{da}{dq}}^{b} f(x)\,dx = \int_{a(q)}^{b} f(x)\,dx - \int_{a(q)}^{a(q)+\delta q\frac{da}{dq}} f(x)\,dx \\
&\approx I(q) + \delta q \frac{dI}{dq}
\end{aligned}
$$



Combining

$$I(q) \equiv \int_{a(q)}^{b(q)} f(x;q)\,dx$$

$$\frac{dI}{dq} \equiv \int_{a(q)}^{b(q)} \frac{\partial f(x;q)}{\partial q}\,dx + \frac{db}{dq} f(b(q);q) - \frac{da}{dq} f(a(q);q)$$

Functions of more than one variable

Covered properly in Lent term

Consider $h(x,y)$

The ***partial derivative*** with respect to x is obtained by treating y as a constant and differentiating what is left

$$\frac{\partial h}{\partial x} \equiv \lim_{\delta x \to 0} \frac{h(x+\delta x, y) - h(x,y)}{\delta x}.$$

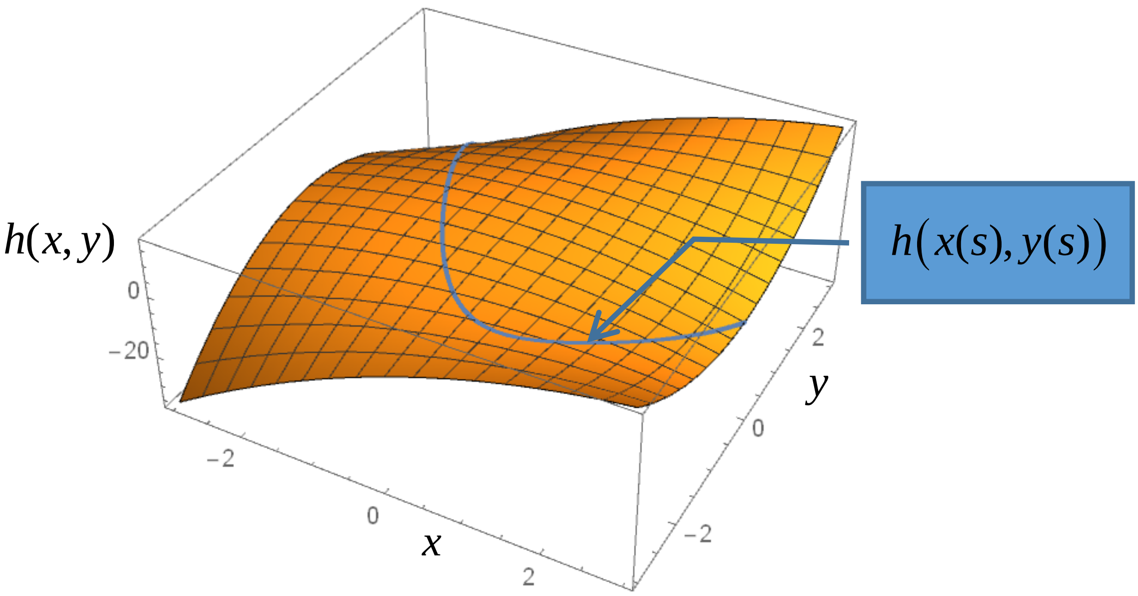
Similarly

$$\frac{\partial h}{\partial y} \equiv \lim_{\delta y \to 0} \frac{h(x, y+\delta y) - h(x,y)}{\delta y}$$

Example: $h(x,y) = 4 - x^2 + (x - \tfrac{1}{2})y^2$

$$\frac{\partial h}{\partial x} = -2x + y^2,$$

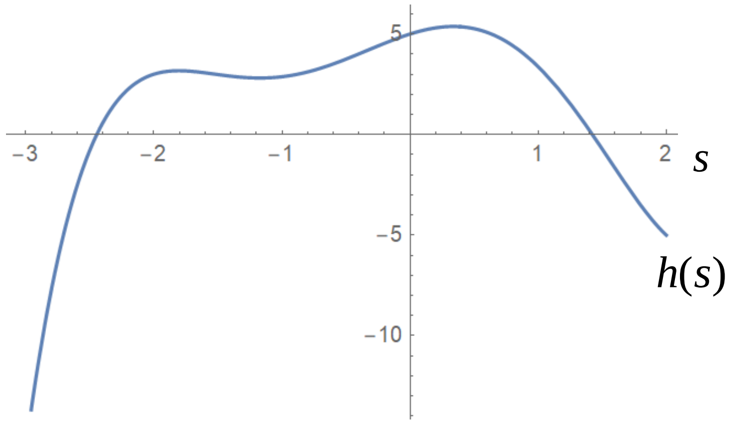
$$\frac{\partial h}{\partial y} = (2x-1)y$$



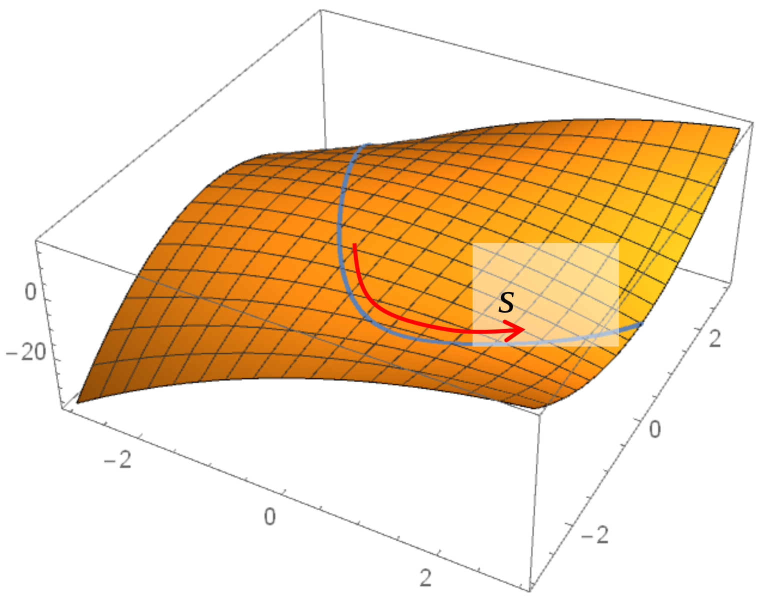
What is $\dfrac{dh}{ds}$ if $x = s+1$ and $y = (s^2-4)/2$?

Could substitute then differentiate:

$$
\begin{aligned}
h(s) = h\big(x(s), y(s)\big) &= 4 - x(s)^2 + (x(s) - \tfrac{1}{2})y(s)^2 \\
&= 4 - (s+1) + (s+1-\tfrac{1}{2})\left(\frac{s^2-4}{2}\right)^2 \\
&= 5 + 2s - 2s^2 - 2s^3 + \tfrac{1}{8}s^4 + \tfrac{1}{4}s^5
\end{aligned}
$$



$$\Rightarrow \qquad \frac{dh}{ds} = 3 - 4s - 6s^2 + \tfrac{1}{2}s^3 + \tfrac{5}{4}s^4$$



Could use the ***chain rule*** for function of more than one variable
$h(x,y) = h\big(x(s), y(s)\big)$

$$\frac{\partial h}{\partial x} = -2x + y^2,$$

$$\frac{\partial h}{\partial y} = (2x-1)y,$$

where $x = s+1$ and $y = \left(s^2-4\right)\big/2$

$$\Rightarrow \qquad \frac{dx}{ds} = 1, \quad \frac{dy}{ds} = s.$$

$$\begin{aligned}
\frac{dh}{ds} &= \frac{\partial h}{\partial x}\frac{dx}{ds} + \frac{\partial h}{\partial y}\frac{dy}{ds} \\
&= \left(-2x+y^2\right)\times 1 + \big((2x-1)y\big)\times s \\
&= \underbrace{\left(-2(s+1) + \left(\tfrac{1}{2}(s^2-4)\right)^2\right)}_{2-2s-2s^2+\frac{1}{4}s^2}\times 1 + \underbrace{\left((2(s+1)-1)\tfrac{1}{2}(s^2-4)\right)}_{-2-4s+\frac{1}{2}s^2+s^3}\times s \\
&= 3 - 4s - 6s^2 + \tfrac{1}{2}s^3 + \tfrac{5}{4}s^4
\end{aligned}$$

Differentiation by parameter

Define primitive $F(x,q) \equiv \int^{x} f(u,q)\,du$

$$\Rightarrow \qquad I(q) = F\big(b(q),q\big) - F\big(a(q),q\big)$$

Chain rule

$$\frac{dI}{dq} = \underbrace{\left.\frac{\partial F}{\partial x}\right|_{x=b(q)}}_{\frac{\partial F}{\partial b}=f(b(q),q)} \frac{db}{dq} + \underbrace{\left.\frac{\partial F}{\partial q}\right|_{x=b(q)}}_{\int^{x}\frac{\partial}{\partial q}[f(u,q)]du} - \underbrace{\left.\frac{\partial F}{\partial x}\right|_{x=a(q)}}_{\frac{\partial F}{\partial a}=f(a(q),q)} \frac{da}{dq} - \underbrace{\left.\frac{\partial F}{\partial q}\right|_{x=a(q)}}_{\int^{x}\frac{\partial}{\partial q}[f(u,q)]du}$$

$$= \int_{a(q)}^{b(q)} \frac{\partial f(x;q)}{\partial q}\,dx + f(b(q);q)\frac{db}{dq} - f(a(q);q)\frac{da}{dq}$$

In this we have also used

Fundamental theorem: $\dfrac{\partial F}{\partial x} = f(x,q)$

Also, taking x constant $\dfrac{\partial F}{\partial q} = \int^{x} \dfrac{\partial}{\partial q}\big[f(u,q)\big]du$

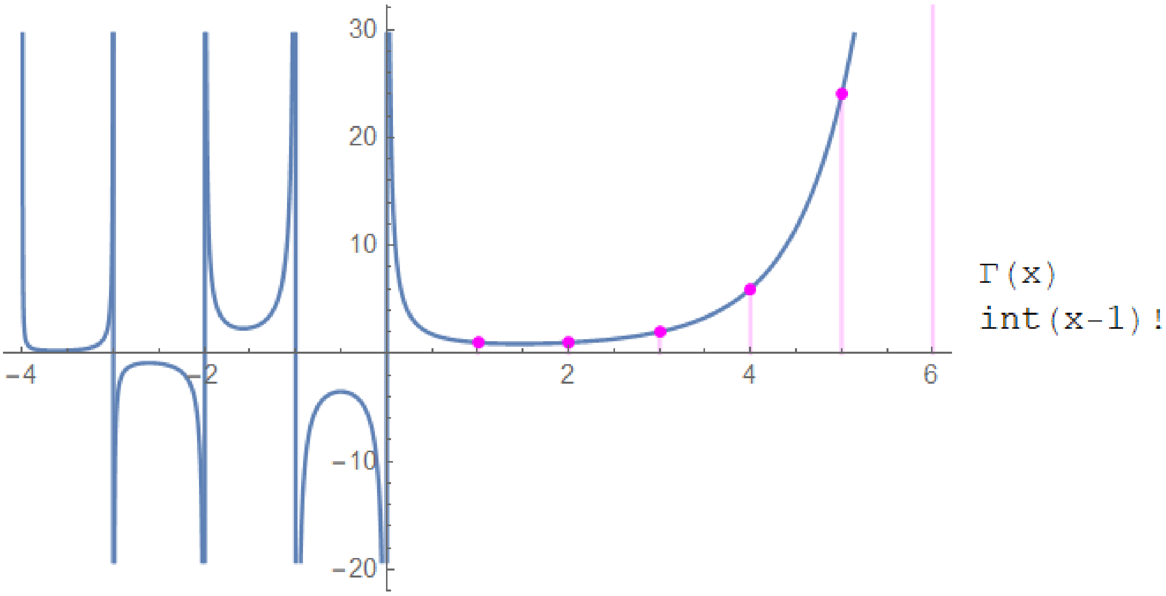
For NST1A, the key is to understand how to use this rather than derive it

Gamma function

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$$

Integrate by parts for integer $x > 0$ or use differentiation of $I(\alpha) = \int_0^\infty e^{-\alpha x}\, dx$ to show relationship with factorials.

For $n \in \mathbb{Z}$, $\Gamma(n) = \begin{cases} (n-1)! & n \geq 1 \\ undefined & otherwise \end{cases}$



Now Lecture 18: Integrals

Recap of Lecture 18: Integration

Schwarz's inequality

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)$$

Riemann integral

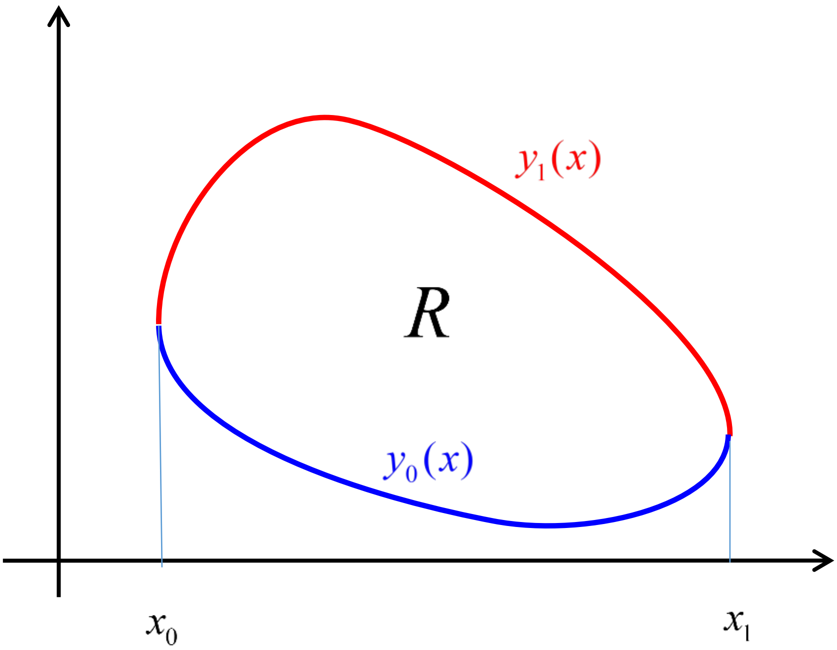
1D $$\int_a^b f(x)\,dx \equiv \lim_{N\to\infty}\left[\sum_{i=1}^{N}(x_i - x_{i-1})\,f(\xi_i)\right]$$

2D $$\int\!\!\int_A h(x,y)\,dA = \lim_{P\to\infty}\left[\sum_{i=1}^{P} h(x_i,y_i)\delta A_i\right]$$

$$\begin{aligned}\int_a^b\int_c^d h(x,y)\,dy\,dx &\equiv \int_a^b\left[\int_c^d h(x,y)\,dy\right]dx \\ &= \int_a^b \lim_{N\to\infty}\left[\sum_{j=1}^{N} h\left(x,\hat{y}_j\right)\delta y\right]dx \\ &= \lim_{M,N\to\infty}\sum_{i=1}^{M}\left[\sum_{j=1}^{N} h\left(\hat{x}_i,\hat{y}_j\right)\delta y\right]\delta x \\ &= \int_c^d\int_a^b h(x,y)\,dx\,dy\end{aligned}$$

Here, we choose $\hat{x}_i, \hat{y}_j$ to lie within the corresponding $\delta A = \delta x \delta y$ elements. For example, $\hat{x}_i = a + (i - \frac{1}{2})\delta x$, $\hat{y}_j = b + (j - \frac{1}{2})\delta y$ and $\delta x = (b-a)/M$, $\delta y = (d-c)/N$.

More complex domains. If R is the region defined by $y_0(x) \le y \le y_1(x)$ for $x_0 \le x \le x_1$

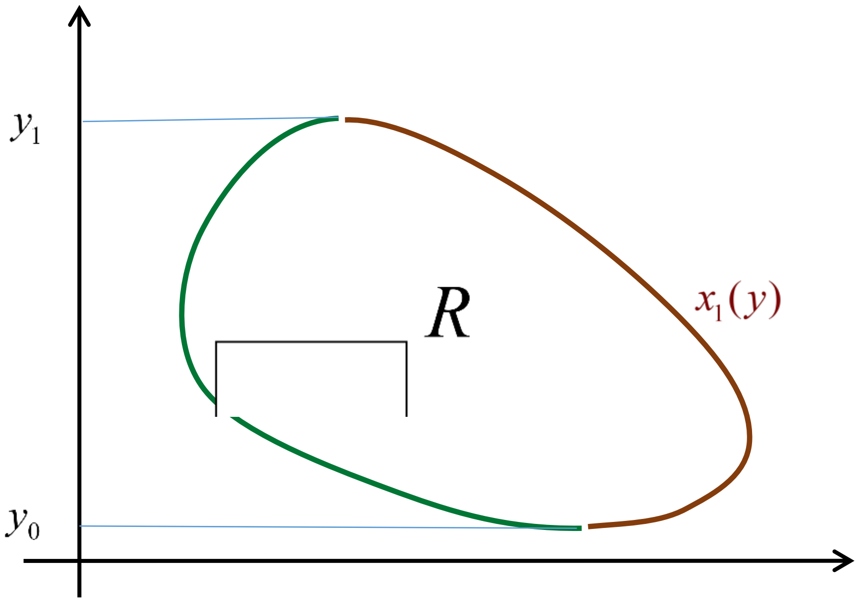


$$\iint_R h(x,y)\,dA = \int_{x_0}^{x_1}\int_{y_0(x)}^{y_1(x)} h(x,y)\,dy\,dx = \int_{x_0}^{x_1} I(x)\,dx$$

where $I(x) = \int_{y_0(x)}^{y_1(x)} h(x,y)\,dy = Y(x,y_1(x)) - Y(x,y_0(x))$

with $Y(x,y) = \int^y h(x,u)\,du$ the primitive of $h(x,y)$ with respect to y ($\partial Y/\partial y = h(x,y)$).

Alternatively, we may be able to define R as $x_0(y) \le x \le x_1(y)$ for $y_0 \le y \le y_1$



$$\iint_R h(x,y)\,dA = \int_{y_0}^{y_1}\int_{x_0(y)}^{x_1(y)} h(x,y)\,dx\,dy = \int_{y_0}^{y_1} J(y)\,dy$$

where $J(y) = \int_{x_0(y)}^{x_1(y)} h(x,y)\,dx = X(x_1(y),y) - X(x_0(y),y)$

with $X(x,y) = \int^x h(s,y)\,ds$ the primitive of $h(x,y)$ with respect to x ($\partial X/\partial x = h(x,y)$).

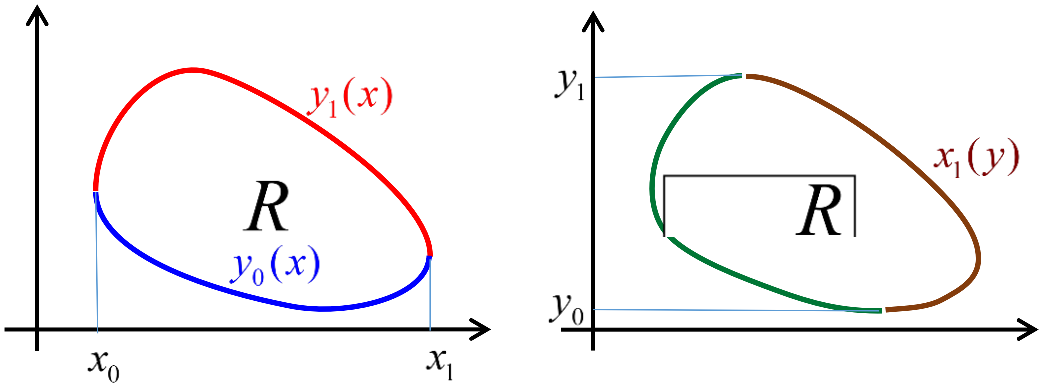
3D $$\iiint_V \rho(x,y,z)\,dV = \lim_{P\to\infty}\left[\sum_{i=1}^{P} \rho(x_i,y_i,z_i)\delta V_i\right]$$

Now Lecture 19: Integration

Recap of Lecture 19: Integration

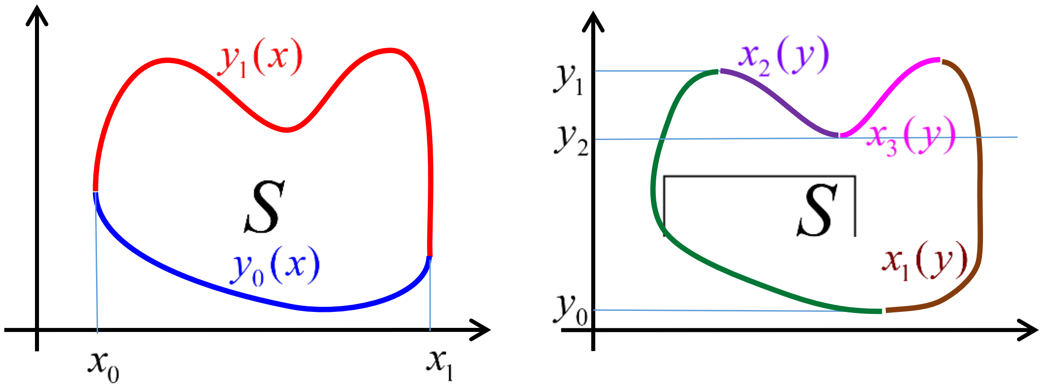
Regions

If region convex



$$\iint_R h(x,y)\,dA = \int_{x_0}^{x_1}\int_{y_0(x)}^{y_1(x)} h(x,y)\,dy\,dx = \int_{y_0}^{y_1}\int_{x_0(y)}^{x_1(y)} h(x,y)\,dx\,dy$$

If region not simply convex, this may be more difficult



$$\begin{aligned}\iint_S h(x,y)\,dA &= \int_{x_0}^{x_1}\int_{y_0(x)}^{y_1(x)} h(x,y)\,dy\,dx \\ &= \int_{y_0}^{y_2}\int_{x_0(y)}^{x_1(y)} h(x,y)\,dx\,dy \\ &\quad + \int_{y_2}^{y_1}\int_{x_0(y)}^{x_2(y)} h(x,y)\,dx\,dy + \int_{y_2}^{y_1}\int_{x_3(y)}^{x_1(y)} h(x,y)\,dx\,dy\end{aligned}$$

Separable integrands

If integrands separable $f(x,y,z)=a(x)b(y)c(z)$ and limits constant

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)\,dx\,dy\,dz = \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} a(x)b(y)c(z)\,dx\,dy\,dz$$

$$= \left[\int_{z_0}^{z_1} c(z)\,dz\right]\left[\int_{y_0}^{y_1} b(y)\,dy\right]\left[\int_{x_0}^{x_1} a(x)\,dx\right]$$

Polar integrals

Circular polar: $dA = r\,dr\,d\theta$

$$\iint_A f(x,y)\,dA = \int_\theta\int_r f(r,\theta)\,r\,dr\,d\theta$$

Cylindrical polar: $dV = r\,dr\,d\theta\,dz$

$$\iiint_V g(x,y,z)\,dV = \int_z\int_\theta\int_r g(r,\theta,z)\,r\,dr\,d\theta\,dz$$

Spherical polar: $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\iiint_V g(x,y,z)\,dV = \int_\phi\int_\theta\int_r g(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

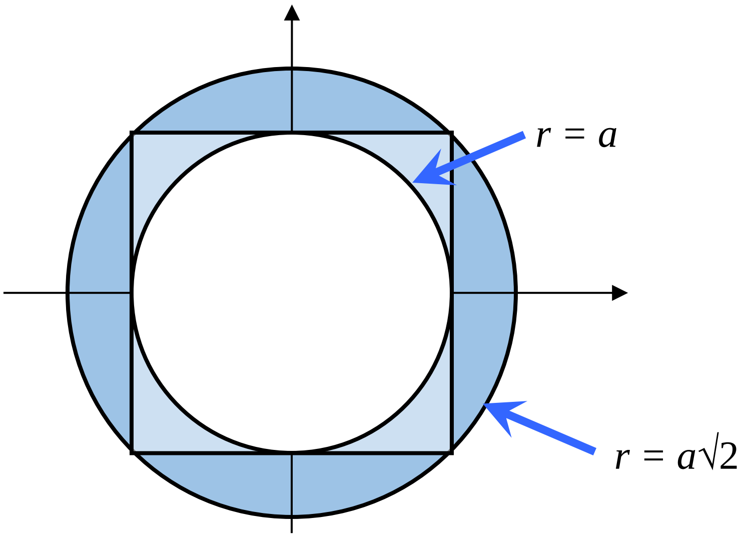
Strategy

1. Choose the coordinate system taking into account the shape of the area A (or volume V) and the form of the integrand $f(x,y)$ (or $g(x,y,z)$) to make the calculation as simple as possible;
2. Determine limits, *e.g.* $y_0(x) \le y \le y_1(x)$ and $x_0 \le x \le x_1$, or $0 \le r \le a$ and $0 \le \theta \le \pi/2$ (three pairs of limits if volume);
3. Rewrite the integrals (including integrand) in terms of the selected coordinate system;
4. Look to see if the integrand is separable and the limits independent of each other;
5. Decide on order in which to integrate;
6. Integrate with respect to one variable at a time, working our way outward through all variables. Each integration eliminates one of the variables.

Gaussian integral

$$I_a = \int_{-a}^{a} e^{-x^2}\, dx$$

$$\begin{aligned} I_a^2 &= \left[\int_{-a}^{a} e^{-x^2}\, dx\right]^2 \\ &= \left[\int_{-a}^{a} e^{-x^2}\, dx\right]\left[\int_{-a}^{a} e^{-y^2}\, dy\right] \\ &= \int_{x=-a}^{x=a}\int_{y=-a}^{y=a} e^{-\left(x^2+y^2\right)}\, dy\, dx \end{aligned}$$



$$\int_{-\pi}^{\pi}\int_{0}^{a} e^{-r^2}\, r\, dr\, d\theta < I_a^2 = \int_{-a}^{a}\int_{-a}^{a} e^{-\left(x^2+y^2\right)}\, dy\, dx < \int_{-\pi}^{\pi}\int_{0}^{\sqrt{2}a} e^{-r^2}\, r\, dr\, d\theta$$

$$J(a) = \underbrace{\int_{-\pi}^{\pi}\int_{a}^{\sqrt{2}a} e^{-r^2}\, r\, dr\, d\theta}_{\text{Integral within annulus}} \le \underbrace{\pi\left(\sqrt{2}-1\right)a^2}_{\text{Area of annulus}}\ \underbrace{e^{-a^2}}_{\text{max value in annulus}}$$

As $a \to \infty$, $e^{-a^2} \to 0$, $ae^{-a^2} \to 0$, $a^2 e^{-a^2} \to 0 \Rightarrow J(a) \to 0$

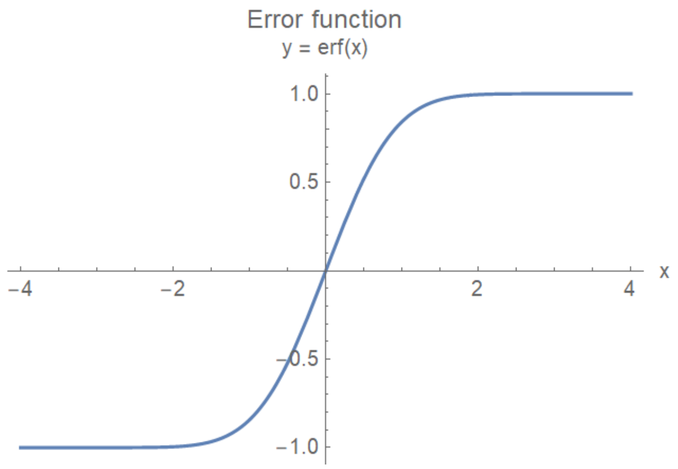
$$\Rightarrow I_a^2 = \int_{x=-a}^{x=a}\int_{y=-a}^{y=a} e^{-\left(x^2+y^2\right)}\, dy\, dx \ \to\ \left[\int_{\theta=-\pi}^{\theta=\pi} d\theta\right]\left[\int_{r=0}^{r=\infty} e^{-r^2}\, r\, dr\right]$$

$$I_\infty^2 = \int_{\theta=-\pi}^{\theta=\pi} d\theta \int_{r=0}^{r=\infty} re^{-r^2}\, dr$$
$$= 2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^\infty$$
$$= \pi$$

$\Rightarrow$
$$I_\infty = \sqrt{\pi}$$

Error function

Error function: $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-x^2}\, dx$; $\displaystyle\lim_{z\to\infty} \operatorname{erf}(z) = 1$.



$$\exp(-x^2) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\, dx$$
$$= \frac{2}{\sqrt{\pi}}\int_0^z \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots\right] dx$$
$$= \frac{2}{\sqrt{\pi}}\left[z - \frac{1}{3}z^3 + \frac{z^5}{5(2!)} - \frac{z^7}{7(3!)} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)(n!)} + \cdots\right]$$

Now Lecture 20: Integration → Probability theory

Lecture 20: Integration → Probability theory

If you encounter the Gaussian integral while answering a question in integration, you can simply quote or make use the result unless you are told explicitly to show/prove that it is true:

$$\int_{-\infty}^{\infty} e^{-r^2}\, dr = \sqrt{\pi}$$

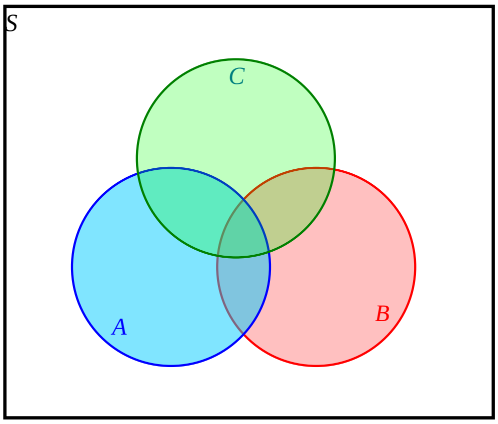
Lecture 20: Integration → Probability theory

8. Probability theory

Sample space: Set of all possible outcomes.

Event: Subset of outcomes.

Ven diagram: A graphical representation of events



If $S = \{x_1, x_2, x_3, \cdots, x_s\}$ is the sample space and $A = \{s_i, s_j, \cdots, s_a\}$ an event (defined by a list outcomes), then all members of A must be contain in S.

Empty set: $\varnothing \equiv \{\}$

Subset: $A \subset S$; $S \subset S$; $\varnothing \subset S$.

Superset: $S \supset A$

Intersection: $A \cap B$ The outcomes common to both events

Union: $A \cup B$ The set of outcomes found in one or both of A, B.

Complement: $\bar{A}$ Outcomes not in event A. $\bar{A} = S - A$. $\bar{\bar{A}} = A$.

Mutually exclusive: $A \cap B = \varnothing$; $A \cap \bar{A} = \varnothing$; Only one event can occur.

Commutative:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative:

$$(A \cup B) \cup C = A \cup (B \cup C);$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distribution

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Negation

$$\overline{A \cap B} = \bar{A} \cup \bar{B}, \quad A \cap B = \overline{\bar{A} \cup \bar{B}},$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}, \quad A \cup B = \overline{\bar{A} \cap \bar{B}}.$$

Related to Boolean algebra ($+ \equiv or$, $\cdot \equiv and$)

$$A \cdot B = \overline{\bar{A} + \bar{B}}; \; A + B = \overline{\bar{A} \cdot \bar{B}}$$

Probability: for N experiments producing n_A occurrences of event A, the probability of A is $P(A) = \lim_{N\to\infty} \frac{n_A}{N}$. The **expected** number of occurrences for N experiments is $N_A = N\,P(A)$.

If all the possible outcomes in sample space $S = \{x_1, x_2, x_3, \cdots, x_s\}$ have the same probability, and event $A \subset S$, then $P(A) = \frac{size(A)}{size(S)}$.

Sample space	$P(S) = 1$
Empty set	$P(\varnothing) = 0$
Event	$0 \le P(A) \le 1$

$$P(A \cap \bar{A}) = 0$$

$$P(A \cup \bar{A}) = 1$$

$$P(\bar{A}) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ -P(A \cap B) - P(B \cap C) - P(C \cap A) \\ +P(A \cap B \cap C) \end{aligned}$$

We can get the last of these from

$$\begin{aligned} P(A \cup B \cup C) &= P((A \cup B) \cup C) \\ &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\ &= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C)) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - \left(P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))\right) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

Now Lecture 21: Probability theory

Lecture 21: Probability theory

Conditional probability: The probability that B occurs, given that A has already occurred is

$$P(B\,|\,A) = \frac{P(A\cap B)}{P(A)}.$$

Often said as "the probability of B given A".

Rearranging gives the probability of both occuring

$$\Rightarrow \qquad P(A\cap B) = P(A)P(B\,|\,A).$$

If the events are ***independent***, then event A has no bearing on the outcome of event B and

$$P(B\,|\,A) = \frac{P(A\cap B)}{P(A)} = P(B),$$

$$\Rightarrow \qquad P(A\cap B) = P(A)P(B\,|\,A) = P(A)P(B).$$

In general, we must consider the possibility that the events are **not *independent***.

Similarly $$P(A\,|\,B) = \frac{P(A\cap B)}{P(B)}$$

$$\Rightarrow \qquad P(A\cap B) = P(B)P(A\,|\,B)$$

$$\Rightarrow \qquad P(A\cap B) = P(B)P(A\,|\,B) = P(A)P(B\,|\,A)$$

Bayes' Theorem: $$P(A\,|\,B) = \frac{P(A)P(B\,|\,A)}{P(B)},$$

Since

$$P(B) = P(B \cap \underbrace{(A \cup \bar{A})}_{S}) = P((B \cap A) \cup (B \cap \bar{A}))$$
$$= P(B \cap A) + P(B \cap \bar{A}) - P(B \cap \underbrace{A \cap \bar{A}}_{\varnothing})$$
$$= P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A})$$

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A})}.$$

Can extend ideas:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)$$

Note: We are multiplying the *conditional* probabilities together. If the events are ***independent***, then

$$P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) = P(A_1)P(A_2)P(A_3),$$

but if they are **not *independent***, then the two products are different.

Permutations: Arranging things **where order matters**. The number of ways of selecting r items from a set of n, where the order matters:

$${}^{n}P_r \equiv {}_{n}P_r \equiv P(n,r) \equiv \frac{n!}{(n-r)!}$$

Now Lecture 22: Probability theory

Lecture 22: Probability theory

Permutations

Order matters

$$^{n}P_{r} \equiv {}_{n}P_{r} \equiv P(n,r) \equiv \frac{n!}{(n-r)!}$$

Combinations

Order does not matter.

The selected permutation $^{n}P_{r} \equiv \dfrac{n!}{(n-r)!}$ can be rearranged $r!$ ways

$$^{n}C_{r} \equiv {}_{n}C_{r} \equiv C(n,r) \equiv \binom{n}{r} \equiv \frac{n!}{(n-r)!r!} \equiv \frac{^{n}P_{r}}{r!}$$

Binary outcomes: an event either happens or it does not happen.

Combinations $\Leftrightarrow$ ***binomial coefficient***

If $P(A) = p$, then for n **independent** experiments,

$$P(A \text{ exactly } r \text{ times out of } n) = {}^{n}C_{r}\, p^{r} (1-p)^{n-r}$$

if it does not matter which of the n experiments yield A.

Can view $^{n}C_{r} \equiv \dfrac{n!}{(n-r)!r!} = \dfrac{n!}{s!r!}$ as the number of ways of arranging $n = r + s$ objects of types R and S if we cannot distinguish between different objects of the same type.

Arrangements

Generalise the binomial arrangements to arrangements of more different types of object.

$$\begin{gathered} r_1 \text{ indistinguishable objects of class } R_1, \\ r_2 \text{ indistinguishable objects of class } R_2, \\ \dots \\ r_k \text{ indistinguishable objects of class } R_k, \end{gathered}$$

giving $n = r_1 + r_2 + \cdots + r_k = \Sigma_{i=1}^k r_i$ objects total. These be arranged

$$\frac{n!}{r_1!r_2!\cdots r_k!} = \frac{\left(\Sigma_{i=1}^k r_i\right)!}{\Pi_{i=1}^k \left(r_i!\right)}$$

distinguishable ways.

Care about the order of everything: $n!$

Don't care about the order of $n-r$ not selected: $\dfrac{n!}{(n-r)!} \equiv {}^nP_r$

Don't care about the order of r selected and $n-r$ not selected

$$\frac{n!}{(n-r)!r!} \equiv {}^nC_r$$

Care about the order of $r+s+t$ but not $u+v+w$;

$$\frac{(r+s+t+u+v+w)!}{u!v!w!}$$

Discrete probability distributions

Let X be a random variable. This is chosen from the set $\{x_0, x_1, x_2, \cdots, x_{n-1}\}$ through a random process with corresponding probabilities $\{p_0, p_1, p_2 \cdots, p_{n-1}\}$.

$$X \in \{x_0, x_1, x_2, \cdots, x_{n-1}\}$$

Only discrete values of X are permitted.

Probability function:

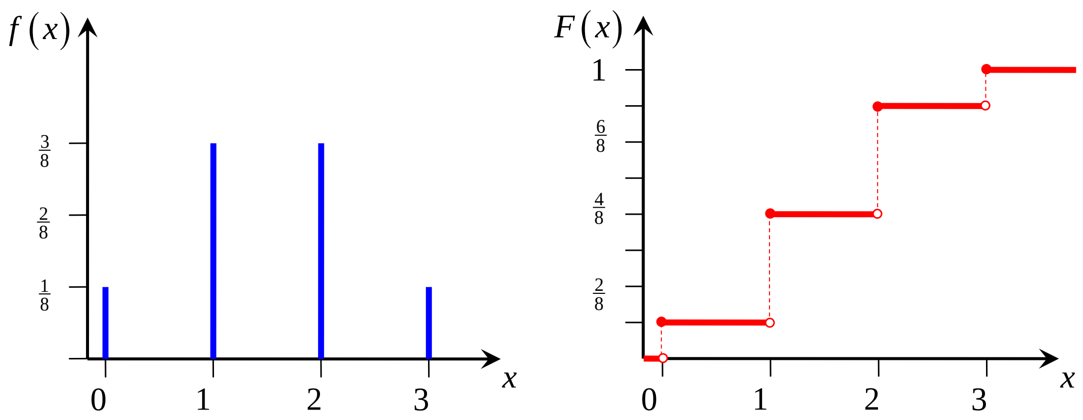
$$P(X = x) = f(x) = \begin{cases} p_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases}$$

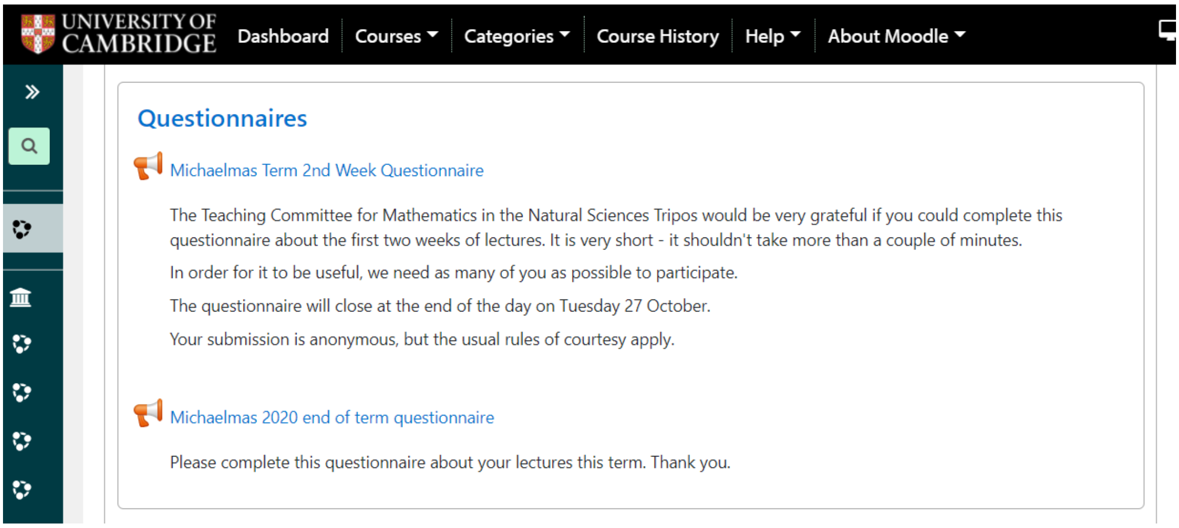
Requires $$\sum_{i=0}^{n-1} f(x_i) = \sum_{x \in \{x_0, x_1, \cdots x_{n-1}\}} f(x) = 1$$

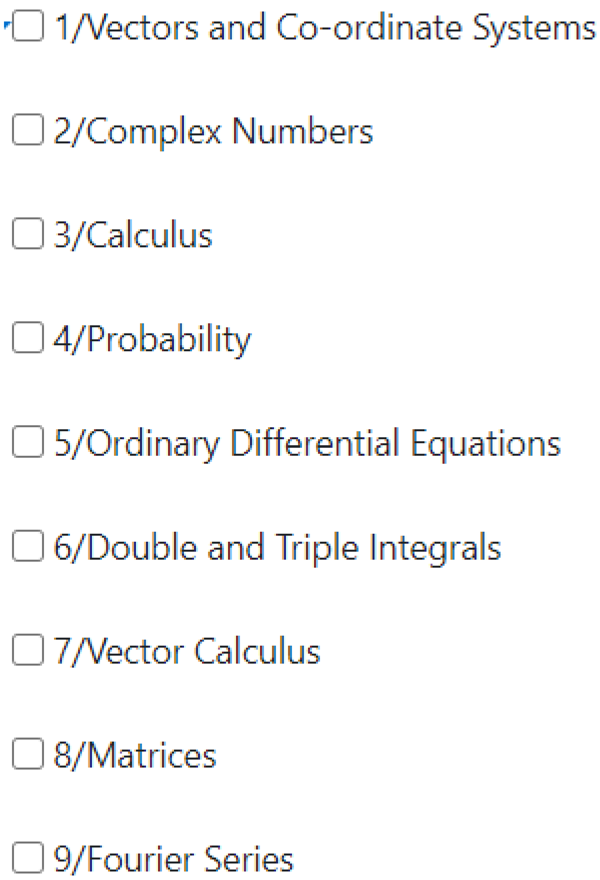
Cumulative probability function:

$$P(X \le x) = F(x) = \sum_{x_i \le x} f(x_i),$$

Note $F(x) = 0$ in the limit $x \to -\infty$, and $F(x) = 1$ in the limit $x \to \infty$.







Now Lecture 23: Probability

Reap of Lecture 23: Probability

The ***mean***, ***expected value*** or ***expectation value*** of a discrete random variable is

$$\mu = \bar{X} = E[X] = \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} x_i f(x_i),$$

and the variance

$$\sigma^2 \equiv E[(X-\mu)^2] = E\left[X^2\right] - \mu^2 = \overline{X^2} - \bar{X}^2$$
$$= \sum_{i=1}^{n} x_i^2 f(x_i) - \left[\sum_{i=1}^{n} x_i f(x_i)\right]^2$$

Relationships

$$E[aX] = aE[X], \quad E[(aX)^2] = a^2 E[X^2],$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[g(X)] = \sum_{i=1}^{n} g(x_i) f(x_i).$$

So

$$E[(X+Y)^2] = E[X^2] + 2E[XY] + E\left[Y^2\right]$$

$$\Rightarrow \qquad \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\operatorname{cov}(X,Y)$$

where the ***covariance****

$$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y] = E[XY] - \mu_X \mu_Y$$

is a measure of how well ***correlated*** the random variables are.

*You do not need to know about the covariance. If the events are independent, then $\operatorname{cov}(X,Y) = 0$.

Binomial distribution

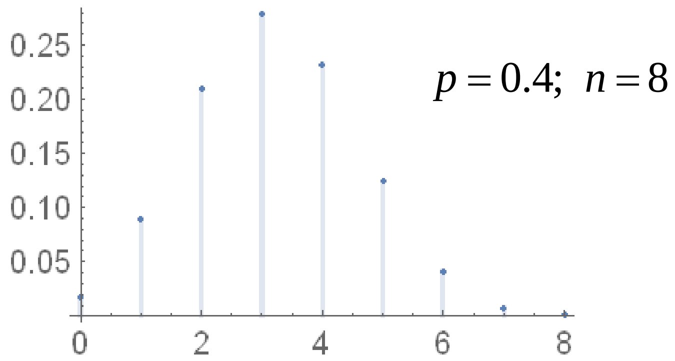
For n discrete, independent events:

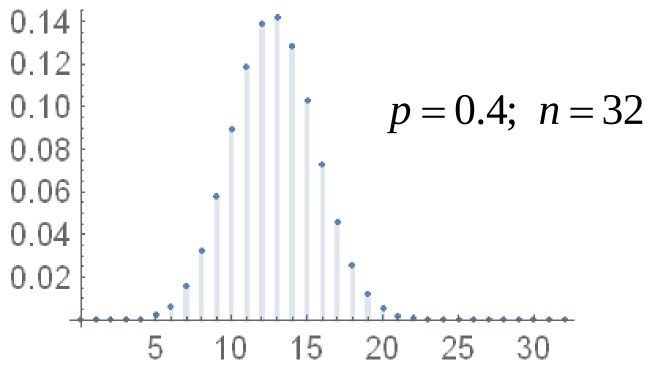
$$f(r) = {}^nC_r\, p^r (1-p)^{n-r}, \quad 0 \le r \le n$$

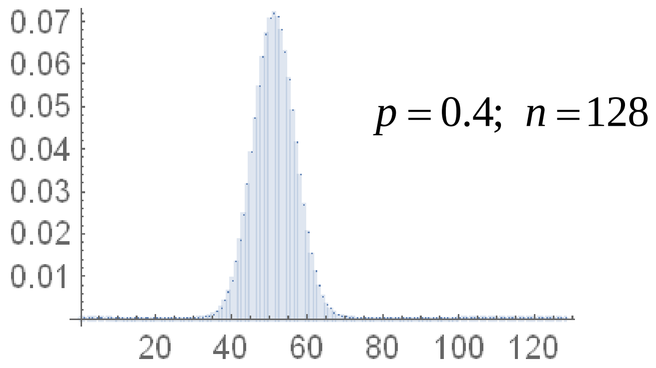
with mean $\mu = E(X) = \sum_{r=0}^{n} r\left[{}^nC_r\, p^r (1-p)^{n-r} \right] = np$

and variance $\sigma^2 = E(X^2) - \left[E(X)\right]^2 = np\left(1-p\right)$

Mean increases as n, standard deviation increases as $\sqrt{n}$







For probability, it is useful to remember

$$(p+q)^n = p^n + np^{n-1}q + \cdots + npq^n - 1 + q$$
$$= \sum_{i=0}^{n} {}^nC_i\, p^i q^{n-i} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} p^k q^{n-k}$$

$$e^x = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

Poisson distribution

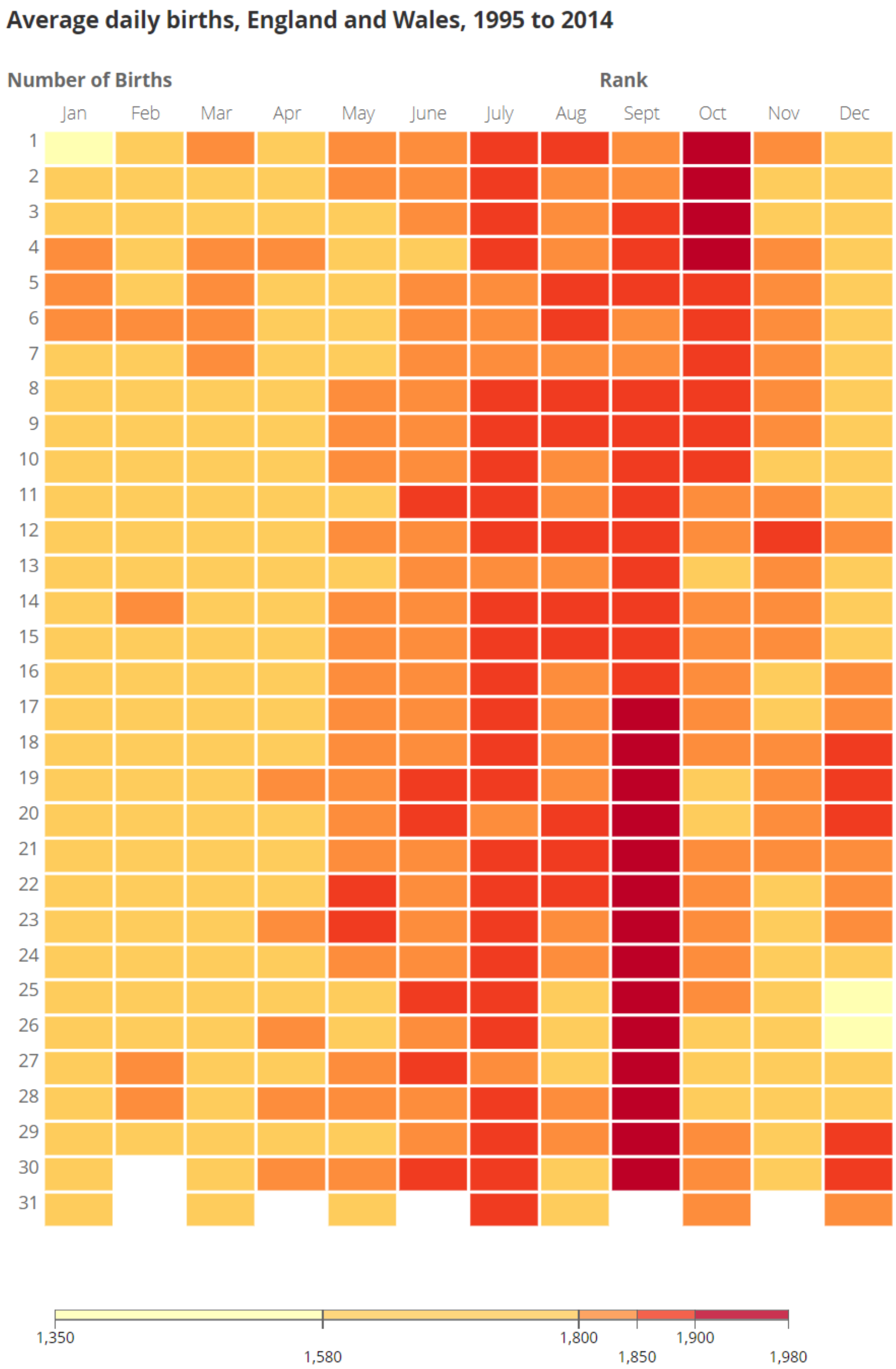
Has no upper limit on the (discrete) outcome:

$$P(X=r) = \frac{\lambda^r \exp(-\lambda)}{r!},$$

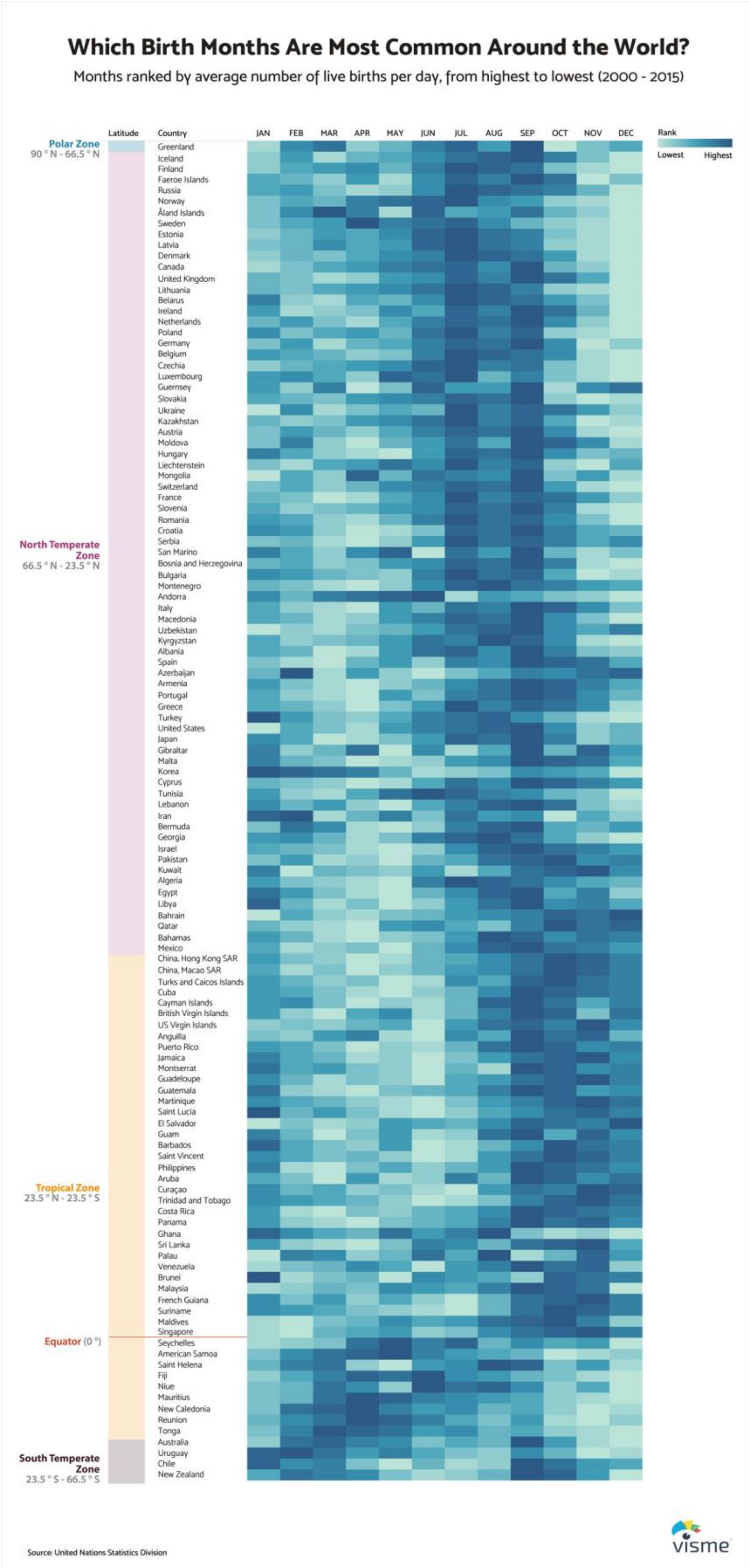
but need finite mean $\mu = \lambda$; variance $\sigma^2 = \lambda$

Poisson distribution is the limit of the binomial distribution when $n \to \infty$, $p \to 0$ but with np remaining finite.

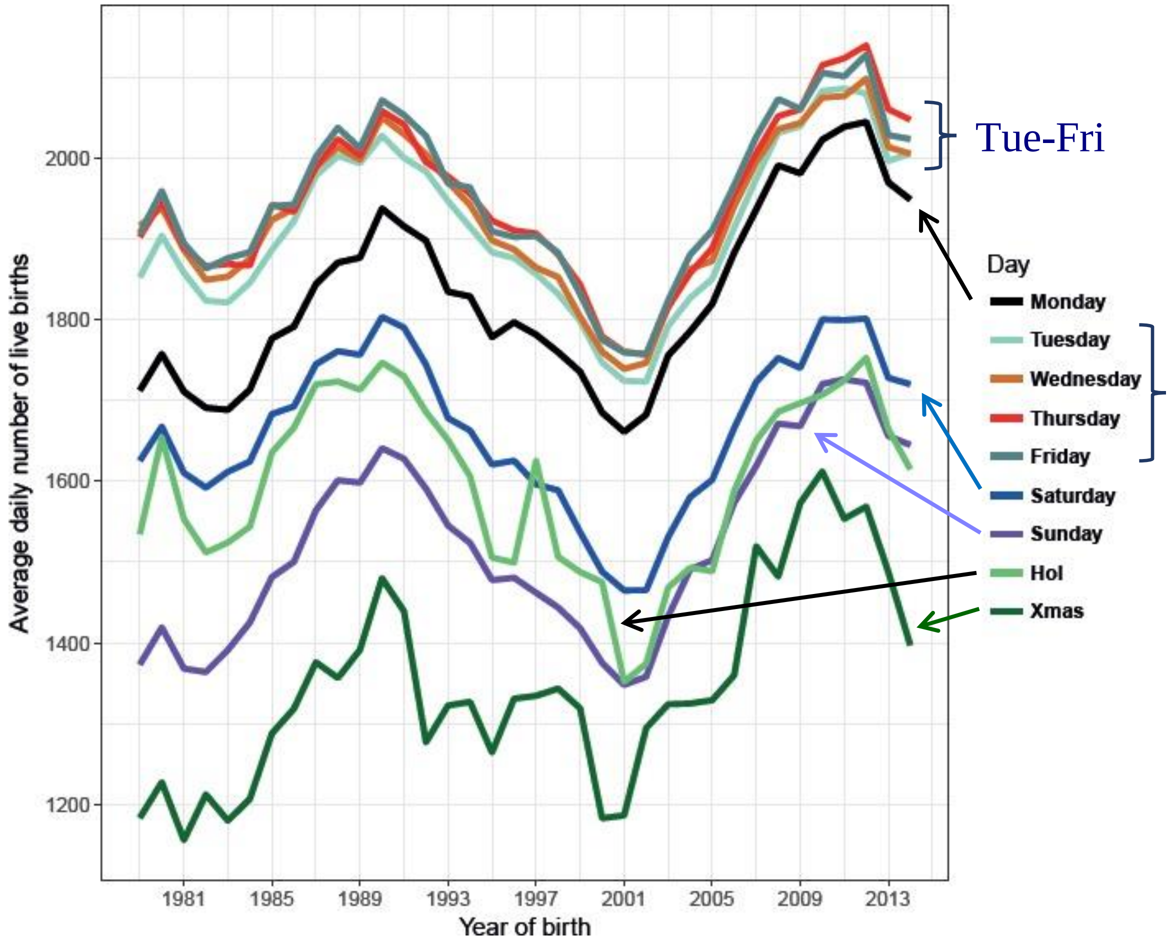
Mean increases as λ, standard deviation increases as $\sqrt{\lambda}$



https://www.ons.gov.uk/peoplepopulationandcommunity/birthsdeathsandmarriages/livebirths/articles/howpopularisyourbirthday/2015-12-18



https://visme.co/blog/most-common-birthday/



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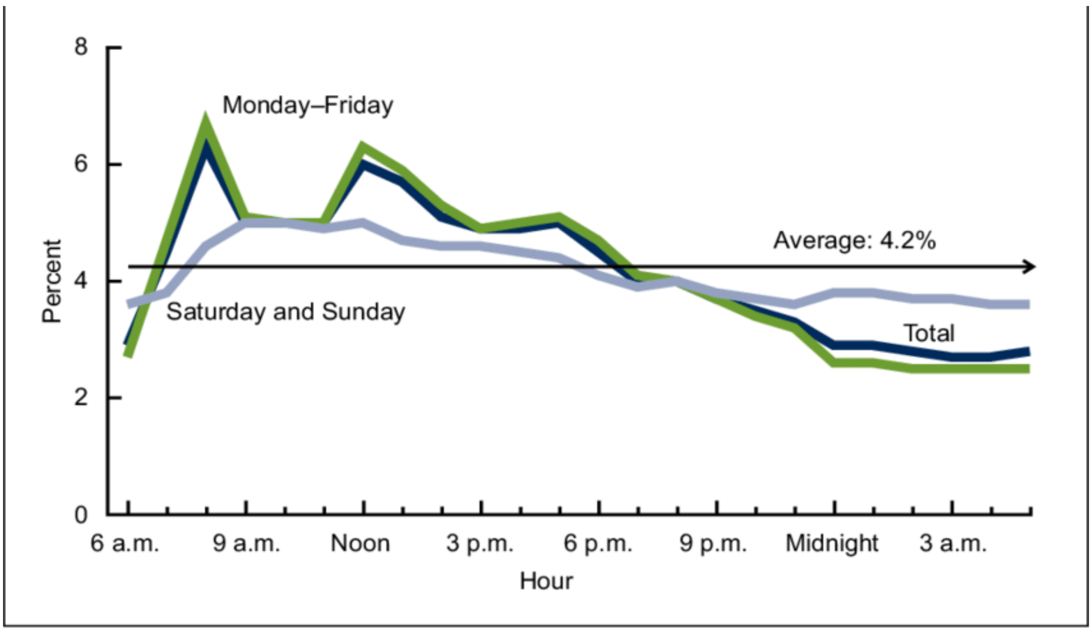


NOTES: The differences in the percent distributions are statistically significant. Access data table for Figure 1 at: http://www.cdc.gov/nchs/data/databriefs/db_200_table.pdf#1.
SOURCE: CDC/NCHS, National Vital Statistics System.

Continuous probability distributions

Random variable $X \in \mathbb{R}$

Probability density function (pdf): $f(x)$ such that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$P(x \le X \le x + dx) = f(x)\,dx \text{ in limit } dx \to 0,$$

so
$$P(\alpha \le X \le \beta) = \int_{\alpha}^{\beta} f(x)\,dx.$$

Cumulative probability function (cpf): $F(x)$ is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx.$$

Primitive of $f(x)$ with $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$.

$$\frac{dF(x)}{dx} = f(x)$$

Mean:
$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

Variance: $\sigma^2 = E\left[X^2\right] - E\left[X\right]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} xf(x)\,dx\right]^2$

Uniform distribution

$$f(x)=\begin{cases}\dfrac{1}{\beta-\alpha} & \text{when } \alpha\le x\le\beta\\ 0 & \text{otherwise}.\end{cases}$$

$$P(X\le x)=\begin{cases}0 & \text{if } x<\alpha\\ \displaystyle\int_\alpha^x \frac{\mathrm{d}x}{\beta-\alpha}=\frac{x-\alpha}{\beta-\alpha} & \text{if } \alpha\le x\le\beta\\ 1 & \text{if } x>\beta.\end{cases}$$

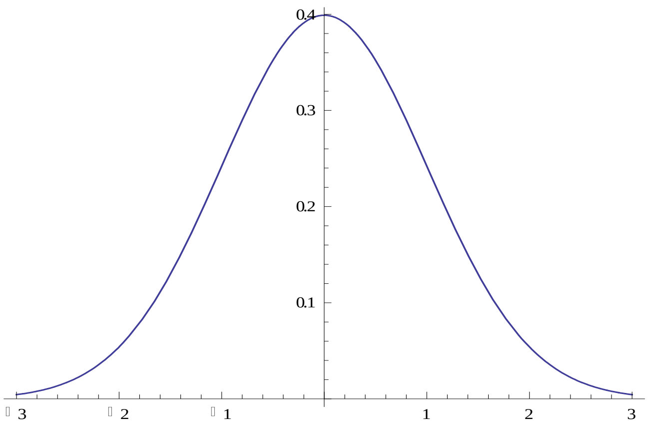
Now Lecture 24: Probability theory

Summary of Lecture 24: Probability theory

Normal distribution or ***Gaussian distribution*** with mean μ and variance σ^2, $N\left(\mu,\sigma^2\right)$:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

$N\left(0,1\right)$:



You will learn how to evaluate the ***Gaussian integral*** in Lent term:

Let
$$I=\int_{-\infty}^{\infty}e^{-x^2}\,dx.$$

Then
$$I^2=\left[\int_{-\infty}^{\infty}e^{-x^2}\,dx\right]^2=\left[\int_{-\infty}^{\infty}e^{-x^2}\,dx\right]\left[\int_{-\infty}^{\infty}e^{-y^2}\,dy\right]$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-x^2}e^{-y^2}\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(x^2+y^2\right)}\,dx\,dy$$

Express $x=r\cos\theta$ and $y=r\sin\theta$ then $dx\,dy=r\,dr\,d\theta$ and

$$I^2=\int_{-\pi}^{\pi}\int_{0}^{\infty}e^{-r^2}\,r\,dr\,d\theta=\left[\int_{-\pi}^{\pi}d\theta\right]\left[\int_{0}^{\infty}e^{-r^2}r\,dr\right]$$
$$=2\pi\int_{0}^{\infty}re^{-r^2}\,dr=2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}=\pi$$

Hence
$$I=\int_{-\infty}^{\infty}\exp(-x^2)\,dx=\sqrt{\pi}$$

You can evaluate $\int_{-\infty}^{\infty} x\exp(-x^2)\,dx$, but there is no need since the integrand is odd and so the result is zero.

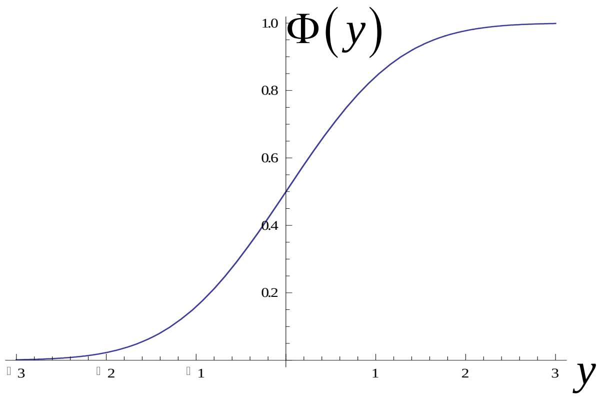
Use integration by parts to evaluate $\int_{-\infty}^{\infty} x^2\exp(-x^2)\,dx = \frac{\sqrt{\pi}}{2}$ using $(x)\left(xe^{-x^2}\right)$ and the Gaussian integral.

Cannot explicitly evaluate cumulative probability function,

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]dy,$$

using elementary functions.

Generally want to convert $N\left(\mu,\sigma^2\right)$ into $N(0,1)$ using the substitution $y = \frac{x-\mu}{\sigma}$. Then $F(x) \to \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi(y)$



and $P(\alpha \le X \le \beta) = \Phi\left(\frac{\beta-\mu}{\sigma}\right) - \Phi\left(\frac{\alpha-\mu}{\sigma}\right)$.

Symmetry shows that $\Phi(y) = 1-\Phi(-y)$.

Central limit theorem states that if we take means over n samples taken from some distribution, $\bar{x} = \frac{1}{n}\sum_{i=0}^{n-1} x_i$, then this mean will be approximately normally distributed about the population mean as n becomes large.

Binomial convergence on Gaussian ($p = 1/2$)