Natural Sciences Part IA Michaelmas term 2020
Magdalene College

Mathematics introductory examples sheet

- Please attempt these problems before you come up. They will be discussed during an examples class (location and time will be announced later), for which you should prepare by spending approximately 12 hours producing formal solutions. The first twelve questions are compulsory. The others are optional, and valuable. You will be required to hand in your completed answers at the latest during your meeting with your Director of Studies on the morning of Monday 5th October. **Please ensure that your work is stapled and you have written your name on the top.** Hand in your work only if you will be attending the examples class.
- Please give attention to your presentation. Write clearly, set out your work so the structure of your solution is plain, and remember that you should give a solution for the marker, not an answer for yourself. Some questions require no prior knowledge; you may know how to do some of the others based on material you have already learnt. If not, try to think carefully about what the problem means and how you might approach it. You are not expected to look up how to do it, especially where the theory is not on A level syllabuses, but to use general problem solving skills. Most of the unfamiliar material will be covered in the Michaelmas term lectures. An outline of the theory and an indication of what to look up, if you wish to do so, is given in the question where appropriate. These questions are targeted at a range of abilities and mathematical backgrounds, and though you should attempt them all, you may not be able to complete every one. If you find that, after making an attempt, you are unable even to start one or more questions, do not worry unduly.
- This piece of work gives a taster of IA NatSci maths, and perhaps an indication of whether you should choose the A course or the B course. If you enjoy these problems and have the requisite background you should consider the B course; if you feel a little overwhelmed or do not have double maths A level you should perhaps consider the A course. It is hoped that you will find the questions challenging and interesting, and that you will work on them independently. Much of the background material is covered in Riley, Hobson and Bence *Mathematical Methods for Physics and Engineering*. You could also consult A level textbooks, either recent or one of the older ones, such as Bostock and Chandler *Further Pure* Mathematics.
- If you are taking Biological Natural Sciences and may like to pursue NST maths, rather than Mathematical Biology, you should attempt these questions. If you ultimately decide not to attend the examples class, please do not hand in your work as marking time is very limited and would need to be given to others.
- Do not spend too long on any one question. You may have failed to spot a straightforward method.
- The 'Mathematics Workbook' is complementary to these problems, testing material with which you should be familiar. There are separate arrangements for handing in your answers; the workbook problems will not be covered in the examples class but may be addressed by your supervisor in your first supervision.

Questions and problems

Compulsory questions

1. I have four hippos: Polly (the largest), Holly, Lyta and Pita (the smallest). Polly is twice as long as Holly, and similarly for Holly and Lyta, and Lyta and Pita. The hippos were all constructed from the same (2D) pattern, scaled equally in both directions using a photocopier. What is the volume ratio between Polly and Pita?
2. Find the exact value of $\cot\frac{\pi}{12}$.
3. a) I take 27 small cubes. The first I number '1' on all its faces, the second '2' and so on. I then build the cubes into a large cube by placing them in sequence so that the bottom layer, viewed from above, looks like

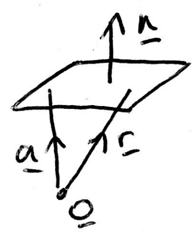
7	8	9
4	5	6
1	2	3

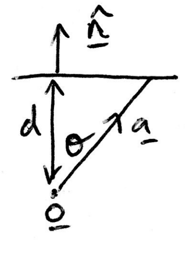
and so on. Draw a net of the large cube, showing the number of each small cube.

b) Again we have a $3\times3\times3$ cube. I remove the central cube from each face, and the central cube from the whole cube. How many small cubes are left?
c) Generalise this to an $n\times n\times n$ cube with an $m\times m$ hole drilled between each pair of opposite faces. Does the exact position of the hole matter?

4. a) Find the equation of the line through (9,4,7) and (3,7,4) in the forms $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$, $\frac{x-x_0}{\lambda}=\frac{y-y_0}{\mu}=\frac{z-z_0}{\nu}$ and $(\boldsymbol{r}-\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{0}$.

b) [Look up: *vector equation of a plane.*]
If, as shown in the diagram, $\boldsymbol{r}$ is the position vector of a general point on the plane, $\boldsymbol{a}$ is the position vector of a specific point on the plane, and $\boldsymbol{n}$ is the normal to the plane, then $\boldsymbol{r}-\boldsymbol{a}$ is perpendicular to $\boldsymbol{n}$. Hence $(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{n}=0 \Rightarrow \boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$. This is the general vector equation of a plane. To find the distance of the plane from the origin, consider the second diagram.





Here d is the perpendicular scalar distance of the plane from the origin, and $\hat{\boldsymbol{n}}$ is the unit normal. From the right-angled triangle we see that $d=|\boldsymbol{a}|\cos\theta=\boldsymbol{a}\cdot\hat{\boldsymbol{n}}$. Thus an alternative form for the equation of the plane is $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}=d$.

Find the equation of the plane through (3,2,6) which is perpendicular to the vector (2,1,-1) in the forms $\boldsymbol{r}.\boldsymbol{n}=d$, $ax+by+cz=d$ and $\boldsymbol{r}=\boldsymbol{r}_0+\lambda\boldsymbol{a}+\mu\boldsymbol{b}$.

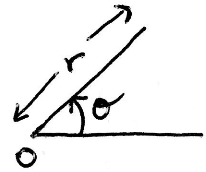
c) Find the position vector of the point where the line in a) intersects the plane. Write your answer in an alternative form by finding its magnitude and direction.

d) A plane passes through the points with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Find the equation of the normal to the plane, and hence its equation in the form $\boldsymbol{r}\cdot\boldsymbol{n}=\lambda$.

5. Prove that $(1-r)(1+r+r^2+\ldots+r^n)=1-r^{n+1}$. What happens as $n\to\infty$ if $-1<r<1$?
Prove that the sum to infinity of a geometric progression whose initial term is a and whose common ratio is r is $\frac{a}{1-r}$. For which range of values of r is this valid?
Calculate the sum to infinity of $153+\frac{153}{2}+\frac{153}{4}+\frac{153}{8}+\ldots$ An arithmetic progression has first term 3.5 and common difference 4. Its sum is equal to that of the sum to infinity you have just calculated. Find the number of terms in the arithmetic progression and its final term.

6. Sketch and identify the following shapes or loci:
i) A plane figure possessing two pairs of parallel sides of equal length
ii) A shape inscribed in a circle[1] and having three edges, one of which is a diameter
iii) A shape inscribed in a circle, every edge of which is the same length as the radius
iv) The area between two concentric circles
v) The locus of the set of points that are equidistant from a fixed point and a fixed line
vi) A solid object having four faces that are equilateral triangles

[Look up: *plane polar coordinates.*] Points are described not in term of x and y coordinates but as a distance, r, from a fixed point and an angle, θ, measured anticlockwise from a fixed line through the point. For example, (1,1) in Cartesians becomes $(\sqrt{2},\pi/4)$ in plane polars, and (-3,4) becomes $(5,\pi-\arctan 4/3)$.



vii), viii), ix), x) In plane polars, $r=a$, $r=\theta$, $\theta=\alpha$, $r=e^{\theta}$
xi) The locus of a set of points, the sum of the distances to two fixed points being constant

1 'inscribed in a circle' means 'having all its vertices on the circle'.

xii) $x^2+y^2+z^2=R^2$. Find the coordinates of the north pole, the south pole and the equator
xiii) $|\boldsymbol{r}-\boldsymbol{a}|=\lambda|\boldsymbol{r}-\boldsymbol{b}|$, $\boldsymbol{a}, \boldsymbol{b}, \lambda$ constants
xiv) Two circles, the tangent to one passing through the centre of the other, and vice versa. Note that the tangents pass throught the intersection points. The result may remind you of a recent astronomical phenomenon
xv) A plane shape consisting of a set of lines of equal length, each at an angle of $+2\pi/n$ to the previous one
xvi) The intersection of the plane $z=3$ with the surface $x^2+y^2=4$ (for all values of z)
xvii) The intersection of $x+y+z=0$ with the surface $x^2+y^2=4$ (for all values of z)
xviii) A solid object formed by drawing a circle and a (non-intersecting) line in a plane, then rotating the circle around the line in three dimensions

7. Define $\cosh x=\dfrac{e^x+e^{-x}}{2}$, $\sinh x=\dfrac{e^x-e^{-x}}{2}$, $\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

Draw a diagram of each function. Give a physical or real-world example of each.

Now solve $\cosh \ln x=\dfrac{19}{8}+\sinh \ln \dfrac{x}{4}$.

8. If you have not studied complex numbers before then leave out this question as it is too big a topic for you to study independently; wait till it is lectured.
a) Complex numbers are the same as plane polar coordinates. Discuss.
b) Draw the complex numbers $1, -\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}-i\dfrac{\sqrt{3}}{2}$ on an Argand diagram. Write them in modulus-argument form.
c) Find $25\exp[i(\pi-\arctan 7/24)]+82\exp(i\arctan 40/9)$.
d) Find the sixth roots of -729 in both modulus-argument and Cartesian form. How do the roots relate to each other?

9. The equation $ax^2+bx+c=0$ has roots α and β. Find equations connecting *a*, *b* and *c* to α and β. Find $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ in terms of *a*, *b* and *c*.
The equation $ax^3+bx^2+cx+d=0$ has roots α, β and γ. By considering the expression $(x-\alpha)(x-\beta)(x-\gamma)$, relate *a*, *b*, *c* and *d* to α, β and γ. Find $\alpha^2+\beta^2+\gamma^2$ and $\alpha^3+\beta^3+\gamma^3$ in terms of *a*, *b* and *c*. Generalise. [You have to decide in what way.]

10. Sketch the curves
a) $y=1-x^2$ b) $y=\cos(\dfrac{\pi}{2}x)$ c) $y=x^2+1$ d) $y=\dfrac{1}{x^2+1}$
e) $y=\dfrac{1}{x^2-1}$ f) $y=\dfrac{1}{1-e^x}$ g) $y=\dfrac{x+3}{(x-1)(x-2)}$ h) $y=(2-|x|)(3+|x|)$
i) $y=e^x\sin x$ j) $x=at^2, y=2at$, a constant

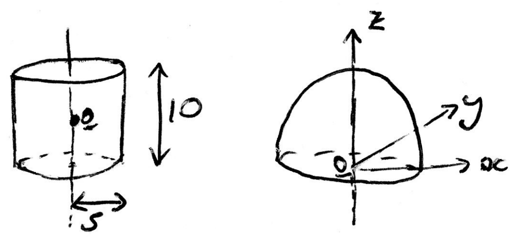
11. Identify the following Greek letters, and give their mathematical meaning or name:
$\mu, \alpha, \theta, \Sigma, \pi, \Delta, \phi, \Omega, \lambda, \Pi, \beta, \Gamma, \omega, \delta, \psi, \nu, \rho, \gamma, \epsilon, \Lambda, \tau, \sigma, \Theta$.

12. Find all solutions of the following equations:
a) $z^4-10z^3+35z^2-50z+24=0$
b) $e^x+35e^{-x}=12$
c) $\dfrac{g^3h^3}{v^6}u^3-3\dfrac{g^2h^2}{v^3}u^2+2ghu=0$ where *g*, *h* and *v* are constants
d) $x^6+6x^5+15x^4+20x^3+15x^2+6x+1=0$
e) $x^8-256=0$

The remaining questions involve problem solving in unfamiliar situations, an extremely valuable skill, and some abstract thinking. The first few of these questions also contain material that is directly relevant to IA. Make sure you spend sufficient time understanding what is being asked and deciding on a strategy before embarking on the more technical part of your solution. Please consider the marker and ensure your solution is complete in itself eg write down the map before workiing out the matrix, or vice versa.

Optional questions

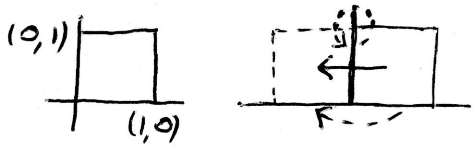
13. Explain what is meant by the unit normal vector to a surface.
Describe the normal to a plane.
Consider a cylinder of radius 5, height 10 and centre the origin. Find the direction of the outward normal to the cylinder at the points a) (-3,4,2), b) (-1,-2,-5) & c) $(5/\sqrt{2}, 5/\sqrt{2}, 5)$.
We now define the vector area as
unit normal x scalar area.
How might you interpret the total vector area for
a) a surface made up of several planar faces, and
b) a curved surface?
Find the vector area of the following:
    a) a sphere of radius R
    b) a disk of radius 2
    c) the open hemisphere whose axis is in the direction of the *z* axis and which is centred on the origin (as shown above)
    d) the triangle with vertices at $(35,6,41)$, $(37,6,41)$, $(36, \sqrt{3}+6, 41)$.
    e) each face of a tetrahedron whose base is the triangle in d) and whose other vertex is at $(36, 6+1/\sqrt{3}, 42)$
    f) the bottom half of the tetrahedron, truncated at half its height



14. Near the point *a*, for *x* small, write $f(a+x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$
By setting $x=0$, find the value of a_0.
Now differentiate both sides with respect to *x* and set $x=0$ to find a_1.
Iterate this procedure to prove that, for sufficiently small x,
$$f(x+a) = f(a) + x f'(a) + \frac{x^2}{2} f''(a) + \ldots + \frac{x^n}{n!} f^{(n)}(a) + \ldots \qquad (1)$$
(1) is called a series expansion, often named after Taylor.

Now set $a=0$. This gives us what is often called a Maclaurin series. Use (1) to prove that
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1} \frac{x^n}{n} + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$
Also write down an expression for $(1+x)^n$ for any real number *n*. Hence, without differentiation, find series expansions for
$$\exp(x^2),\ e^x \sin x,\ \log(1+x)/(1+x),\ \log(6+x),\ \exp(\sin x),\ \exp(\cos x).$$

15. Consider the unit square in $\mathbb{R}^2$, defined by the unit basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.



Let A be a reflection in the y axis, as illustrated in the diagram on the right.

Write down the vectors $A\begin{pmatrix}1\\0\end{pmatrix}$ and $A\begin{pmatrix}0\\1\end{pmatrix}$.

Now place them side by side to form the 2 by 2 transformation matrix $\left(A\begin{pmatrix}1\\0\end{pmatrix}\ A\begin{pmatrix}0\\1\end{pmatrix}\right)$.

Find the transformation matrices for the following maps:

- B is a rotation anticlockwise by $\pi/2$
- C is a rotation anticlockwise by an angle θ
- D is a reflection in the line $y = x$
- E is a reflection in the axis $x\sin\theta = y\cos\theta$
- I is the map 'do nothing'
- F is the map that scales each direction by a factor of 2
- G scales the x direction by a and the y direction by b

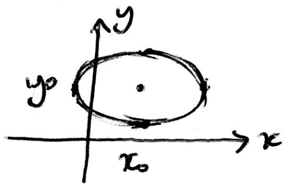
Now describe the effect of each of the following matrices by considering their action on the unit basis vectors, or on a unit square:

$H=\begin{pmatrix}2 & 0\\0 & 1/2\end{pmatrix};\ J=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix};\ K=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix};\ L=\begin{pmatrix}2 & 0\\0 & -1/2\end{pmatrix};\ M=J^2;\ N=B^2;$
$O=B^3;\ P=B^4;\ Q=E^{2n}.$

The determinant of a 2 by 2 matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ is $ad-bc$. This represents the area of the unit square under the transformation. Find the determinant for some of your transformations both algebraically and geometrically, and comment on your results.

16. Conics have been truncated from A level syllabuses over the last 20 years, and no doubt before. This question introduces you to some of the theory.

a) Find the equation of the circle through (-1,5), (-6,4) and (-1,-1) in the form $ax^2+by^2+cx+dy+f=0$, where $a,b,c,d,f\in\mathbb{Z}$.



b) The standard equation for an ellipse takes the form $\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}=1$ (2), where, if we assume $a>b$, a is known as the semi-major axis and b as the semi-minor axis. Find the equation of the landscape-oriented ellipse centred on the origin with semi-major axis of length 5 and semi-minor axis of length 4. Give its parametric equations ie write x and y in terms of a single parameter. (You may find it helpful to consider first what your answer would be for a circle.)

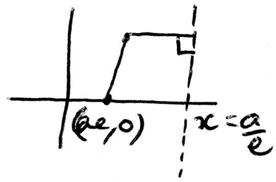
c) By drawing a diagram, or otherwise, find where the circle in a) intersects the ellipse in b).

d) A general conic may be written in the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$. The eccentricity e is defined as $e^2=1-\dfrac{b^2}{a^2}$ if $e<1$ ie for an ellipse, and $e^2=1+\dfrac{b^2}{a^2}$ if $e>1$ ie for a hyperbola.

Further, the eccentricity, and the conic, may be defined geometrically using

distance from fixed point = e x distance from fixed line.

Which conic has zero eccentricity, and which has an eccentricity of 1? If, as shown in the diagram, the fixed point (called the focus) is at $(ae, 0)$ and the fixed line (the directrix) is at $x = a/e$ for $0 < e < 1$, identify the conic. Note that if $e > 1$ then the conic is called a hyperbola.



e) A glasses prescription may be viewed as the correction required to convert the ellipse which one sees into the point which one should see. The sphere measurement acts equally in all directions, while the cylinder corrects at the specified angle (measured clockwise from the positive *y* axis) by a distance equal to the semi-major axis minus the semi-minor axis. Draw diagrams showing the original ellipses for

i) sphere = -3, cylinder =-1 at 70 degrees

ii) sphere = -4, cylinder =+1 at -20 degrees. Comment.

f) A conic is expressed in polar coordinates as $r = \dfrac{C}{1 + e\cos\theta}$, where *C* is a constant. Put this in Cartesian form (2).

One of Kepler's laws of planetary motion states that planets move in elliptical paths with the sun at a focus. Where is the sun?

g) If the eccentricity is greater than or equal to 1, the conic does not form a closed path. Draw a diagram of the conic given by $\dfrac{(x-2)^2}{9} - (y+4)^2 = 1$, and identify it. What are the equations of the asymptotes?

17. Describe the effect on the function $f(x)$ of the following transformations, where *a* is a constant: $f(x+a)$; $f(x-a)$; $f(ax)$; $f(x/a)$; $f(x)+a$; $f(-x)$; $f(a-x)$.
Define the word 'convoluted'. Write down a convoluted sentence.

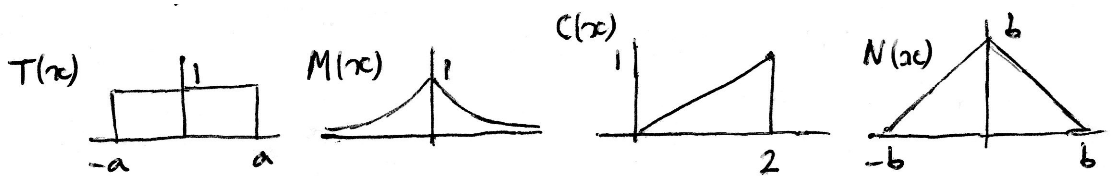
We define a mathematical convolution as $f * g(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$.

Let a train function be given by $T(x) = 1,\ -a \le x \le a$, and zero otherwise.

Let a mountain function be given by $M(x) = e^{x}, x < 0;\ M(x) = e^{-x}, x \ge 0$.

Let a crescendo function be given by $C(x) = x/2,\ 0 \le x \le 2$, and zero otherwise.

Let a Nelson's hat function be given by $N(x) = b + x, -b \le x < 0;\ b - x, 0 \le x \le b;$ and zero otherwise.



Find the convolution of

i) a train with a train

ii) a train with a mountain

iii) a mountain with a mountain

iv) a crescendo with a crescendo

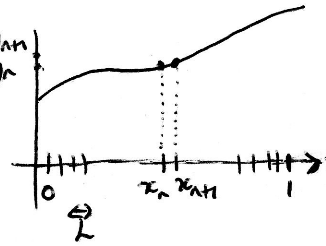
Now define the Fourier transform of a function $f(x)$ to be $\tilde{F}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$.

You should assume that $f(x)$ tends to zero sufficiently rapidly as *x* tends to +/- infinity for the integrals to converge, and that it is possible to perform integrals involving complex numbers in the same way as for real integrals.

Calculate the Fourier transforms of a train function, a mountain function and a Nelson's hat function.
What do you notice about the Fourier transforms of the train function and the Nelson's hat function for a particular value of *b*? How does this relate to your convolution from part i)?

Note: a way of thinking about Fourier transforms is that we move from position space to a new space, perhaps momentum space. Consider what you have done today in terms of your position. Now describe the same events in terms of your momentum. Compare the important features of each. How might this be used to solve differential equations?

18. Let the interval [0,1] be divided into *M* line segments each of length *h*. We approximate $y(x)$ by a series of points $\{y_n, n=0,1,2,\ldots,M\}$.

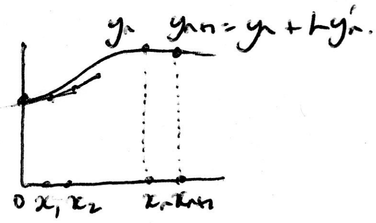


a) Fill in the blanks ⟦ ⟧ in the following equations:

$$x_n=⟦\ ⟧;\quad y_n=⟦\ ⟧;\qquad y^{⟦\ ⟧}=\frac{y_{n+1}-y_n}{h};$$

$$y^{⟦\ ⟧}=\frac{y_{n+1}-y_{n-1}}{⟦\ ⟧};\qquad y^{⟦\ ⟧}=\frac{y_{n+1}-2y_n+y_{n-1}}{⟦\ ⟧^2}.$$

b) Consider the differential equation $y'=-\frac{y}{1+x}$ with $y(0)=1$. We will use Euler's method: $y_{n+1}=y_n+h\,y'_n$. Suggest a possible problem with Euler's method. For the given differential equation, find an expression for y_{n+1} in terms of y_n, and hence an expression for y_n in closed form. Solve the differential equation exactly, and compare your solution with the approximation for y_n in the limit $h\to 0$, $n\to\infty$ with $nh=x$ fixed. Show that in this limit the exact solution is recovered.



19. An equivalence relation on a set of elements (which may be numbers, geometrical shapes, people, etc) possesses the three properties:

it is reflexive ie each element is related to itself: $aRa \quad \forall\, a$
it is symmetric ie $aRb \Rightarrow bRa \quad \forall\, a, b$
it is transitive ie $aRb, bRc \Rightarrow aRc$

Example: consider the set of integers. We say that *a* is related to *b* (*aRb*) if *a*+*b* is an integer.
aRa if *a*+*a* is an integer, which is true because *a* is an integer, and the sum of two integers is another integer. So the relation is reflexive
If *aRb* then this means that, by the definition of the relation, *a*+*b* is an integer. Hence *b*+*a* is an integer, so *bRa*. Thus the relation is symmetric.
To check transitivity, we begin with *aRb* if *a*+*b* is an integer and *bRc* if *b*+*c* is an integer. We wish to show *aRc* ie *a*+*c* is an integer. However, let *a*=1/3, *b*=2/3, *c*=4/3. Then *a*+*b*=1, *b*+*c*=2 but *a*+*c*=5/3 which is not an integer. So the relation is not transitive and hence is not an equivalence relation.
However, if instead we say that, again for the integers, *aRb* if *a*-*b* is an integer, then the relation is
reflexive because *aRa* if *a*-*a*=0 is an integer, which is true for all *a*.
symmetric because if *a*-*b* is an integer then so is *b*-*a*=-(*a*-*b*).
transitive because if *a*-*b* is an integer, and *b*-*c* is an integer, then so is *a*-*c*=(*a*-*b*)-(*b*-*c*).
Thus the relation is an equivalence relation.

Which of the following are equivalence relations? A 'yes' or 'no' answer is inadequate; you should give a careful proof.

a) *aRb* if *a* is a sibling of *b*
b) *aRb* if *a* has the same birthday as *b*
c) *aRb* if *a* beats *b* in 'scissors, paper, stone' (or 'rock, paper, scissors' if you prefer)
d) For polygons, *aRb* if *a* has the same number of edges as *b*
e) *aRb* if *a* is greater than *b*
f) *aRb* if *a* is less than or equal to *b*
g) *aRb* if *a* divides *b*
h) *aRb* if *a* and *b* have a common factor (obviously 1 does not count)
i) *aRb* if *a* is approximately equal to *b*
j) *aRb* if *a* and *b* are congruent in arithmetic modulo *n* for some *n*

20. Suppose that you are given the second order differential equation

$$-[p(x)y'(x)]'+q(x)y(x)=\lambda w(x)y(x),$$

where $p(x)\geq 0$ and $w(x)\geq 0$ on a real interval $[a,b]$. An expression for the eigenvalue λ, subject to suitable boundary conditions eg $y(a)=0$, $y(b)=0$ or $p(a)=0$, $p(b)=0$, is then

$$\lambda = \int_a^b p y'^2+q y^2 dx \ / \int_a^b w y^2 dx .$$

i) Using an integrating factor, convert the differential equation $-y''-4y'-4y=n^2\pi^2 y$, where *n* is an integer, into the standard form given above.

ii) For $-(1+x^2)y''-2xy'=\lambda y$, $y(1)=0$, $y(-1)=0$, write down an expression for the eigenvalue. Find an approximation for the eigenvalue using $y_1=1-x^2$. Find another using $y_2=\cos(\pi x/2)$. You may use the result $\int_{-1}^{1} x^2\sin^2(\frac{\pi x}{2})dx=\frac{1}{3}+\frac{2}{\pi^2}$. Comment on how you might improve your approximation.

iii) Show that $y=\exp(-x^2/2)$ exactly solves $-\frac{d^2y}{dx^2}+(x^2-1)y=\lambda y$, and find the value of λ. How does *y* behave as $x\rightarrow\pm\infty$? These are the boundary conditions for this problem.

We now define the approximate solution $u(x)=b(a^2-x^2)$, $|x|\leq a$; $u(x)=0$, $|x|>a$, where the constants *a* and *b* can be adjusted to improve the approximation. Draw a diagram showing the exact and approximate solutions. Explain why the corresponding approximation for λ, using $u(x)$, might be expected to be a reasonable one, and show that $\lambda(a,b)=\frac{5}{2a^2}-1+\frac{a^2}{7}$ independent of *b*. By minimising this function over *a*, find the best possible approximation for λ and compare it with the exact value.

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