Natural Sciences Tripos, Part IA
Mathematical Methods II, Course B
Example Sheet 3
Scalar and Vector Fields / Fourier Series

Dr S. M. Colwell

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This example sheet is for course B. Starred questions are intended as extras; do them if you have time, but try to complete the others first. Please communicate any errors to `smc1@cam.ac.uk`.

Skills section

Questions in this section are intended to give practice in routine calculations.

S1. For each of the following functions $f(\boldsymbol{x}) = f(x, y, z)$, evaluate $\boldsymbol{\nabla} f$.

(a) $f = xyz$ (b) $f = \exp[-1/(x+y+z)]$

(c) $f = \exp(-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2)$ (d) $f = r$

(e) $f = r^{-1}$ (f) $f = F(r)$

[In (d), (e) and (f), $r = |\boldsymbol{x}| = (x^2 + y^2 + z^2)^{1/2}$.]

S2. Write down parametrizations of the following curves:

(a) a circle in the (x, y) plane, with radius 1 and centre $(1, 1, 0)$

(b) a straight line from $(-1, 0, 0)$ to $(1, 1, 1)$

(c) a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$

S3. A vector field $\boldsymbol{F}(\boldsymbol{x})$ has components $(x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$. Evaluate

(i) $\operatorname{div} \boldsymbol{F}$

(ii) $\operatorname{curl} \boldsymbol{F}$

S4. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be fixed vectors. Show that $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{x}) = \boldsymbol{a}$. Evaluate the divergence and the curl of each of the following vector functions of $\boldsymbol{x}$:

$$\boldsymbol{x}\,, \qquad \boldsymbol{a}(\boldsymbol{x} \cdot \boldsymbol{b})\,, \qquad \boldsymbol{a} \times \boldsymbol{x}\,, \qquad \boldsymbol{x}/|\boldsymbol{x}|^3\,.$$

S5. (**Fourier series**.) Evaluate the following integrals (where m is an integer):

(a)	$\int_0^{2\pi} x\cos(mx)\,\mathrm{d}x$	(b)	$\int_0^{2\pi} x\sin(mx)\,\mathrm{d}x$
(c)	$\int_0^{1} \mathrm{e}^{ax}\cos(2\pi mx)\,\mathrm{d}x$	(d)	$\int_0^{1} \mathrm{e}^{ax}\sin(2\pi mx)\,\mathrm{d}x$
(e)	$\int_{-\pi}^{\pi} \cos^3 x\cos(mx)\,\mathrm{d}x$	(f)	$\int_{-\pi}^{\pi} \cos^3 x\sin(mx)\,\mathrm{d}x$

Standard questions

Scalar and vector fields

6. For the function $f(x, y, z) = \ln(x^2 + y^2) + z$, find $\boldsymbol{\nabla} f$.

   Consider a cylinder of radius 5 whose axis is along the z axis.

   (i) What is the rate of change of $f(x, y, z)$ in the direction normal to the cylinder at the point $(3, -4, 4)$?

   (ii) What is the rate of change of $f(x, y, z)$ in the direction $\boldsymbol{m} = \boldsymbol{i}+2\boldsymbol{j}$ at the same point, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors parallel to the x and y axes, respectively?

7. Find a normal vector to the surface $xz + z^2 - xy^2 = 5$ at the point $(1, 1, 2)$. Deduce the equation of the tangent plane at this point.

8. Obtain the equation of the plane that is tangent to the surface $z = 3x^2y\sin(\pi x/2)$ at the point $x = y = 1$. Take North to be the direction $(0, 1, 0)$ and East to be the direction $(1, 0, 0)$. In which direction will a marble roll if placed on the surface at $x = 1, y = \frac{1}{2}$?

9. Evaluate

$$\int_\Gamma [P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y]$$

   where $P = -x^2y$, $Q = xy^2$ and Γ is the closed curve consisting of the semicircle $x^2 + y^2 = a^2$, $y > 0$, and the segment $(-a, a)$ of the x axis, described anticlockwise. Verify that this is equal to

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\,,$$

   where D is the plane surface enclosed by Γ. (This illustrates Green's theorem in the plane.)

10. Find a function $f(y)$ such that

$$\int_\Gamma [f(y)\,\mathrm{d}x + x\cos y\,\mathrm{d}y] = 0$$

   for all closed contours Γ in the (x, y) plane.

   [*Hint: When is the integrand an exact differential?*]

11. The work done by a force $\boldsymbol{F}$ acting on a particle that moves along a curve C is defined as the line integral

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} \, .$$

(i) When $\boldsymbol{F} = \boldsymbol{c} \times \boldsymbol{v}$, where $\boldsymbol{c}$ is a constant vector and $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is the velocity of the particle, show that the work done is zero.

(ii) A particle moves along the helical path given by

$$x = \cos t \, , \qquad y = \sin t \, , \qquad z = t \, .$$

Calculate the work W done in the time interval $0 \le t \le \pi$ by each of the forces $\boldsymbol{F}$ given by (a) $y\,\boldsymbol{i} - x\,\boldsymbol{j} - \boldsymbol{k}$ and (b) $x\,\boldsymbol{i} + y\,\boldsymbol{j}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors parallel to the x, y and z axes, respectively.

12. Write down a condition obeyed by a conservative vector field $\boldsymbol{F} = (P(x,y), Q(x,y))$. Do the following choices for P and Q yield conservative fields?

(a) $P = x^2 y + y, \quad Q = xy^2 + x$

(b) $P = y\,\mathrm{e}^{xy} + 2x + y, \quad Q = x\,\mathrm{e}^{xy} + x$

In the case that $\boldsymbol{F}$ is conservative, find a function $f(x,y)$ such that $\boldsymbol{F} = \boldsymbol{\nabla} f$.

Consider the following curves, each joining $(0,0)$ to $(1,1)$:

$C_1: \quad x = t\,, \quad y = t \quad (0 \le t \le 1)$

$C_2: \quad x = 0\,, \quad y = t \quad (0 \le t \le 1)\,; \quad x = t\,, \quad y = 1 \quad (0 \le t \le 1)$

Evaluate the integrals $\int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y$ and $\int_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y$ for each of the fields (a) and (b) above. Comment on the results.

13. Consider the vector field

$$\boldsymbol{F} = (4x^3 z + 2x, z^2 - 2y, x^4 + 2yz) \, .$$

Evaluate the line integral $\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}$ along

(i) the sequence of straight-line paths joining $(0,0,0)$ to $(0,0,1)$ to $(0,1,1)$ to $(1,1,1)$;

(ii) the straight line joining $(0,0,0)$ to $(1,1,1)$, given parametrically by

$$x = y = z = t \quad (0 \le t \le 1) \, .$$

Show that $\boldsymbol{F}$ is conservative by finding a function $f(x,y,z)$ such that $\boldsymbol{F} = \boldsymbol{\nabla} f$.

14. Let $\boldsymbol{E} = (-y\,\mathrm{e}^{-2t}, x\,\mathrm{e}^{-2t}, 0)$ and $\boldsymbol{B} = (0, 0, \mathrm{e}^{-2t})$ be two vector fields that depend on time t. Evaluate $\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ and $\int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{x}$, where S is the circular disc $x^2 + y^2 < 1$, $z = 0$ and C is the curve bounding S. Show that

$$\int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{x} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

15. Calculate $\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$, where $\boldsymbol{F} = \alpha x^3\,\boldsymbol{i} + \beta y^3\,\boldsymbol{j} + \gamma z^3\,\boldsymbol{k}$ (with α, β and γ constants):

    (i) over the surface of a sphere of radius a, centred on the origin

    (ii) over the surface of a cylinder of radius a and height $2h$, centred on the origin with its axis along the z axis

    [*Hint: You can reduce the amount of algebra by exploiting the symmetries of the sphere and the cylinder.*]

16. A cube is defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

    (i) Evaluate the surface integral
$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$
    over the surface of the cube, where $\boldsymbol{F} = (x^2 + ay^2, 3xy, 6z)$ and a is a constant.

    (ii) Evaluate also
$$\iiint f\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
    over the volume of the same cube, where $f = bx + 6$ and b is a constant.

    For what values of a and b do these integrals have the same value?

17. Let $\boldsymbol{u}$ be the vector field $\boldsymbol{u} = Q\boldsymbol{x}/(4\pi\epsilon_0 r^3)$ in three dimensions, where $\boldsymbol{x}$ is the position vector, $r = |\boldsymbol{x}|$, and Q and ϵ_0 are constants. Show that
$$\int_S \boldsymbol{u} \cdot \boldsymbol{n}\,\mathrm{d}S = Q/\epsilon_0\,,$$
    where S is a sphere of radius a centred on the origin, and $\boldsymbol{n}$ is a unit normal vector pointing radially outwards from the sphere. (This is Gauss's law for a point charge.)

18. (a) Find all values of a and b for which the integrals in Question 16 are equal by using the divergence theorem.

    (b) For $\boldsymbol{E}$ and $\boldsymbol{B}$ as in Question 14 show that
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,,$$
    and apply Stokes's theorem to deduce the equality of the integrals.

19. State the divergence theorem.

    Let $\boldsymbol{F}(\boldsymbol{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the (open) surface
$$1 - z = x^2 + y^2\,, \qquad 0 \le z \le 1\,.$$
    Evaluate $\int_S \boldsymbol{F} \cdot \boldsymbol{n}\,\mathrm{d}S$, where $\boldsymbol{n}$ is the unit normal on S having a positive component in the z direction.

    [*Hint: Construct a closed surface including S and use the divergence theorem.*]

20. State Stokes's theorem, and verify it for the hemispherical surface $r = 1$, $z \geq 0$ and the vector field $\boldsymbol{A}(\boldsymbol{x}) = (y, -x, z)$.

Fourier series

21. Show that the functions 1, x, $\frac{1}{2}(3x^2 - 1)$ and $\frac{1}{2}(5x^3 - 3x)$ are orthogonal on the interval $[-1, 1]$.

22. Show that
$$\int_0^a \sin(mx)\sin(nx)\,\mathrm{d}x = 0$$
if m and n are positive integers with $m \neq n$ and $a = k\pi$, where k is an integer (i.e. a is an integer multiple of half-wavelengths of the fundamental mode, $\sin x$).

23. Show that, for $m \neq n$ (where m and n are positive integers),
$$\int_{-T}^{T} \sin(m\pi\theta/T)\sin(n\pi\theta/T)\,\mathrm{d}\theta = 0\,.$$
Find the value of the integral when $m = n$.

24. Write down the Fourier series on the interval $[-\pi, \pi]$ (with period 2π) for (i) $\sin 2\theta$ and (ii) $\cos^2\theta$. Find the Fourier series for $\sin^3\theta$.

25. Find the Fourier series for the function that equals $|x|$ when $-l \leq x \leq l$ and is periodic with period $2l$. If this series is integrated, what will the resulting series sum to? If the series is differentiated, what will the resulting series sum to (if anything)? Give sketches.

    * Is the rate of convergence of each series roughly what you might expect from considering the smoothness properties of the function?

26. (i) Find the Fourier series with period 2π that converges to e^x for $-\pi < x < \pi$. To what does it converge when $x = \pi$ and $x = -\pi$?

    (ii) * Show that any function $f(x)$ can be written (in terms of $f(x)$ and $f(-x)$) as the sum of an even function and an odd function. Deduce from part (i) the Fourier series for $\cosh x$ and $\sinh x$ with the same range and periodicity.

27. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ and let $g(x) = \sum_{m=1}^{\infty} B_m \sin mx$. Show that

$$\int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x = \pi \sum_{n=1}^{\infty} b_n B_n\,.$$

Without detailed calculation, give the corresponding result when

$$f(x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,.$$

28. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. What can be said about the coefficients a_n and b_n if $f(x)$ has the following symmetries?

(a) $f(x) = f(-x)$ (b) $f(x) = -f(-x)$ (c) $f(x) = f(\pi - x)$
(d) $f(x) = -f(\pi - x)$ (e) $f(x) = f(\pi/2 + x)$ (f) $f(x) = f(\pi/2 - x)$
(g) $f(x) = f(2x)$ (h) $f(x) = f(-x) = f(\pi/2 - x)$

29*. Assume that

$$x^2 = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{inx} \qquad \text{for} \quad -\pi \le x \le \pi\,,$$

for some complex coefficients c_n. By multiplying both sides by e^{-imx} and integrating, find the coefficients c_n. Write down the (real) Fourier series for x^2 on $[-\pi, \pi]$ with period 2π.

30*. Let $f(x) = x$ for $0 \le x < \pi$. Sketch: (i) the *odd* function, and (ii) the *even* function that are periodic with period 2π and equal to $f(x)$ for $0 \le x < \pi$.

Show that the function $f(x)$ can be represented for $0 \le x < \pi$ either as

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

or as

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}\,.$$