

Natural Sciences Tripos, Part IA  
Mathematical Methods II, Course B  
**Example Sheet 2**  
**Functions of More than One Variable**

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This example sheet is for course B. Starred questions are intended as extras; do them if you have time, but try to complete the others first. Please communicate any errors to [smc1@cam.ac.uk](mailto:smc1@cam.ac.uk).

### Skills section

Questions in this section are intended to give practice in routine calculations.

S1. For each of the following functions, evaluate the partial derivatives  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ ,  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial y^2}$ ,

and verify that  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$ .

- (a)  $f = x^3 - 2x^2y + 3xy^2 - 4y^3$       (b)  $f = \exp(-x^2y^2)$   
(c)  $f = \exp(-x/y)$       (d)  $f = \sin(x + y)$   
(e)  $f = 1/(x^2 + xy + 2y^2)$

S2. For each of the following functions, write out the differential expression  $df = P(x, y) dx + Q(x, y) dy$  showing the explicit form of the functions  $P(x, y)$  and  $Q(x, y)$ .

- (a)  $f = \exp[-1/(x + y)]$       (b)  $f = \sinh x / \sinh y$   
(c)  $f = (x^2 + y^2)^{1/2}$       (d)  $f = \arctan(y/x)$   
(e)  $f = x^y$

S3. For each of the functions in question S1, find the locations of any stationary points. (You need not determine the type of the stationary points.)

## Standard questions

### Partial differentiation

4. For the function

$$f(x, y) = x(y^2 + 2y - 1),$$

find the components of the vector  $(\partial f/\partial x, \partial f/\partial y)$ , known as the gradient vector, at the points  $(-1, 0)$ ,  $(1, 0)$ ,  $(-1, 1)$  and  $(1, 1)$ . Make a sketch showing the directions of the gradient vector at these points.

5. The period  $T$  of a simple pendulum of length  $\ell$  swinging in a gravitational field  $g$  is given by

$$T = 2\pi(\ell/g)^{1/2}.$$

Estimate the percentage error in a measurement of  $g$  resulting from

- (a) a 0.1 % error in measuring  $\ell$ ,
- (b) a 0.1 % error in measuring  $T$ .

[*Hint: take logs before differentiating!*]

6. For  $f(x, y) = \exp(-xy)$ , find  $(\partial f/\partial x)_y$  and  $(\partial f/\partial y)_x$ . Then find  $(\partial f/\partial r)_\phi$  and  $(\partial f/\partial \phi)_r$

- (i) by using the chain rule,
- (ii) by first expressing  $f$  in terms of polar coordinates  $(r, \phi)$ ,

and check that the two methods give the same results.

7. If  $xyz + x^3 + y^4 + z^5 = 0$  (an implicit equation for any of the variables  $x, y, z$  in terms of the other two), find

$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y,$$

and show that their product is  $-1$ .

8. Van der Waals's equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

is an early (and in many ways a remarkably successful) attempt to represent the relation between the pressure  $p$ , volume  $V$  and temperature  $T$  of a real gas. ( $R$ ,  $a$  and  $b$  are constants for a given mass of the gas.) Find expressions for  $(\partial p/\partial V)_T$ ,  $(\partial V/\partial T)_p$  and  $(\partial T/\partial p)_V$ , and verify that their product is  $-1$ .

9.  $f(x, y)$  is a scalar function of position on the  $(x, y)$  plane. Position may also be specified by Cartesian coordinates  $u, v$  which are referred to axes rotated by an angle  $\theta$  from the  $x$  and  $y$  axes. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2},$$

i.e. the two-dimensional  $\nabla^2$  operator is invariant under a rotation of axes.

## Exact differentials

10. Determine whether the following differentials  $P dx + Q dy$  are exact. For those that are exact, find a function  $f$  such that  $df = P dx + Q dy$ . For those that are not, try to find an integrating factor  $\mu(x)$  or  $\mu(y)$ . In each case, find the general solution of the differential equation  $P dx + Q dy = 0$ .

(a)  $y dx + x dy$

(b)  $y dx + x^2 dy$

(c)  $(x + y) dx + (x - y) dy$

(d)  $(\cosh x \cos y + \cosh y \cos x) dx - (\sinh x \sin y - \sinh y \sin x) dy$

(e)  $(\cos x - \sin x) dx + (\sin x + \cos x) dy$

(f)  $(x dy - y dx)/(x^2 + y^2)$

(You may wish to discuss with your supervisor why case (f) presents difficulties.)

11. The enthalpy of a gas is defined by  $H = U + pV$ , where  $U$  satisfies  $dU = T dS - p dV$ . Determine a relationship between the differentials of  $H$ ,  $S$  and  $p$ . Hence show that

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S.$$

By regarding  $U$  as a function of  $p$  and  $V$  and considering two expressions for  $\partial^2 U / \partial p \partial V$ , show that

$$\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial T}{\partial p}\right)_V - \left(\frac{\partial S}{\partial p}\right)_V \left(\frac{\partial T}{\partial V}\right)_p = 1.$$

12. Given that

$$dU = T dS - p dV,$$

find a function  $G$  such that

$$dG = V dp - S dT.$$

Hence show that

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p.$$

13. The pressure  $p$  can be considered as a function of the variables  $V$  and  $T$  or as a function of the variables  $V$  and  $S$ .

(i) Find an expression for

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \quad \text{in terms of} \quad \left(\frac{\partial S}{\partial V}\right)_T \quad \text{and} \quad \left(\frac{\partial S}{\partial p}\right)_V.$$

(ii) Hence, using  $T dS = dU + p dV$ , show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S = \left(\frac{\partial(pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right]. \quad (\star)$$

$$\left[ \text{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p} \left(\frac{\partial p}{\partial V}\right)_T \cdot \right]$$

(iii) For one mole of an ideal gas, you may assume that  $U = C_v T$ ,  $pV = RT$ , and  $pV^\gamma$  depends only on  $S$ , where  $C_v$ ,  $R$  and  $\gamma$  are constants.

Use equation  $(\star)$  to find an expression for  $\gamma$  in terms of  $C_v$  and  $R$ .

(iv) What is the value of  $\gamma$  for a monatomic gas for which  $C_v = \frac{3}{2}R$ ?

### Stationary values

14. The height  $h$  of each point  $(x, y)$  of an area of land is given by

$$h(x, y) = \frac{a(x + y)}{x^2 + y^2 + a^2},$$

where  $a$  is a positive constant. Find the locations and heights of the highest and lowest points of the terrain, and also those along the  $x$  and  $y$  axes. Sketch a map of the region by showing contours of constant  $h$  in the  $(x, y)$  plane.

15. (i) Find the stationary points of the function

$$z = (x^2 - y^2) e^{-x^2 - y^2}.$$

(ii) Find the contours on which  $z = 0$  and examine the behaviour of  $z$  on the  $x$  and  $y$  axes. Hence, or otherwise, determine the character of the stationary points. Sketch the contours.

16. For each of the functions

(a)  $(x^2 + y^2 + 1)^{-1}$

(b)  $\sin x \sin y \quad (0 < x < \pi, 0 < y < \pi)$

(c)  $(xy - y) \exp(2x - x^2 - y^2) \quad [\text{more difficult}]$

find the stationary points and determine their character.

## Conditional stationary values

17. Find, using Lagrange multipliers, the stationary values subject to the constraint  $x^2 + y^2 = 1$  of

(a)

$$xy^2$$

(b)

$$e^{-xy}$$

and the points at which they occur. Check your answers by applying the constraint using the substitution  $x = \cos \theta$ ,  $y = \sin \theta$ .

18. Show that the largest volume  $V$  of any cuboid inscribed inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is  $8abc/\sqrt{27}$ .

19. The area  $A$  of a triangle with sides  $a$ ,  $b$  and  $c$  is given by  $A = \sqrt{s(s-a)(s-b)(s-c)}$ , where  $s = \frac{1}{2}(a+b+c)$ . Show, using Lagrange multipliers, that of all triangles of given perimeter  $2s$ , the triangle of largest area is equilateral.

Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given perimeter.

- 20\*. Show that the maximum value,  $r$ , of  $(x^2 + y^2 + z^2)^{1/2}$ , subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \ell x + my + nz = 0,$$

satisfies the equation

$$\frac{\ell^2 a^2}{a^2 - r^2} + \frac{m^2 b^2}{b^2 - r^2} + \frac{n^2 c^2}{c^2 - r^2} = 0.$$

Interpret the problem geometrically and state the geometrical significance of one of the Lagrange multipliers.

## Boltzmann distribution

21. Let

$$W = \prod_{s=1}^N \frac{(g_s - 1 + n_s)!}{(g_s - 1)! n_s!} \quad (\dagger)$$

and let the numbers  $n_s$  be subject to the conditions

$$\sum_{s=1}^N n_s = n, \quad \sum_{s=1}^N n_s E_s = E.$$

If *both*  $g_s$  and  $n_s$  are assumed to be large, so that Stirling's approximation  $\ln n! \approx n \ln n - n$  may be applied, show that  $\ln W$ , considered as a function of the numbers  $n_s$ , is greatest when

$$n_s = g_s [e^{\beta(E_s - \mu)} - 1]^{-1}, \quad (\ddagger)$$

where  $\beta$  and  $-\beta\mu$  are Lagrange multipliers.

Equation  $(\ddagger)$  is in fact the Bose–Einstein distribution and follows from the formula  $(\dagger)$  which takes into account the effects of quantum indistinguishability for bosons. However you do not need to know any quantum mechanics to do this question!

For fermions the formula for  $W$  turns out to be

$$W = \prod_{s=1}^N \frac{g_s!}{n_s! (g_s - n_s)!}.$$

Find the value of  $n_s$  for which this expression for  $W$  is maximized subject to the same conditions. This is the Fermi–Dirac distribution.

22\*. A finite container is filled with a monatomic ideal gas of constant internal energy (total kinetic energy). It can be shown that the degeneracy of the kinetic energy states is proportional to  $E^{1/2}$ .

The number of particles  $n$  of kinetic energy  $E$  is therefore distributed as

$$n \propto E^{1/2} e^{-\beta E - \alpha},$$

where  $\alpha$  and  $\beta$  are constants. Determine the most probable value and the expected value of the kinetic energy of a molecule.