Natural Sciences Tripos, Part IA
Mathematical Methods II, Course B
Example Sheet 1
Ordinary Differential Equations

Dr S. M. Colwell

Lent Term 2021

This example sheet is for course B. Please communicate any errors to `smc1@cam.ac.uk`.

Skills section

Questions in this section are intended to give practice in routine calculations. On this sheet the skills questions start off with integration, as revision from Michaelmas Term, since integration is an important mathematical operation in the solution of ordinary differential equations.

S1. Find the indefinite integral $\int f(x)\,\mathrm{d}x$ when $f(x)$ is given by:

(a) $(1+x)^{1/4}$ (b) $\dfrac{2+x}{(1+x)^2}$

(c) $2x(3+x^2)$ (d) $2x\sin(x^2)$

(e) $\cot x$ (f) $x^2\sin x$ [*Hint: Integration by parts*]

(g) $\sin^3 x$ [*Hint: Trig. identity*] (h) $\dfrac{x}{(1-x)(2-x)}$ [*Hint: Partial fractions*]

(i) $\dfrac{1}{1+x^2}$ [*Hint: Substitution*] (j) $\sin\sqrt{1-x}$ [*Hint: Substitution*]

S2. Verify that the following are solutions of the corresponding differential equations where in each case $c, d, \ldots$ are arbitrary constants.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,,$ $\qquad y = \frac{1}{2}x^2 + c\,.$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y\,,$ $\qquad y = c\,\mathrm{e}^x\,.$

(c) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0\,,$ $\qquad y = c\,\mathrm{e}^{-4x} + d\,\mathrm{e}^{-x}\,.$

(d) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0\,,$ $\qquad y = cx^{-4} + dx^{-1}\,.$

(e) $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4} - y = 0\,,$ $\qquad y = c\sin x + d\cos x + f\,\mathrm{e}^x + g\,\mathrm{e}^{-x}\,.$

(f) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 4y^3 = \dfrac{8}{x^6}\,,$ $\qquad y = x^{-2}\,.$

S3. Solve by separation of variables:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^3}{(y+1)^2}\,.$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4y}{x(y-3)}\,.$$

S4. Solve by the use of integrating factors:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 4x\,.$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (2-3x^2)x^{-3}y = 1\,.$$

Standard questions

5. Solve by change of variables and separation:

$$(x+y+1)^2\,\frac{\mathrm{d}y}{\mathrm{d}x} + (x+y+1)^2 + x^3 = 0\,.$$

6. Solve by change of variables and the use of integrating factors:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y = xy^5\,.$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2(\cos x - \sin x)\,.$$

7. Solve the homogeneous equation

$$(y-x)\frac{\mathrm{d}y}{\mathrm{d}x}+(2x+3y)=0\,.$$

8. Solve:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)\,.$$

(b)

$$(\ln y-x)\frac{\mathrm{d}y}{\mathrm{d}x}-y\ln y=0\,.$$

[Find a substitution that simplifies the equation.]

(c)

$$xy\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2+y^2+x)=0\,.$$

9. Solve the following differential equations subject to the initial conditions $y(0)=0$, $y'(0)=1$.

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=0\,.$$

(b)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+n^2\right)y=0\,.$$

(c)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\frac{\mathrm{d}}{\mathrm{d}x}+4\right)y=0\,.$$

(d)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+9\right)y=18\,.$$

(e)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-3\frac{\mathrm{d}}{\mathrm{d}x}+2\right)y=\mathrm{e}^{5x}\,.$$

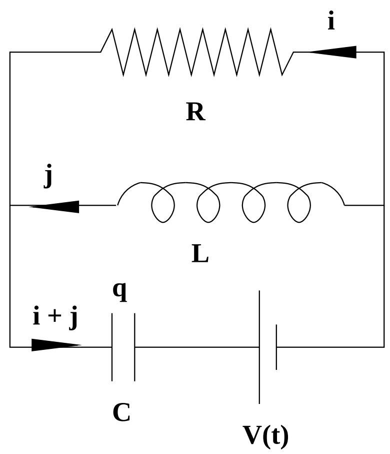
For parts (f) and (g) find the general solution:

(f)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+y=0\,.$$

(g)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+y=\mathrm{e}^{2x}+\mathrm{e}^{x}\,.$$



10. A 20th century electrical circuit consists of a resistor (resistance R), an inductor (inductance L), a capacitor (capacitance C) and a power supply (voltage $V(t)$) in the arrangement shown in the diagram below. A current $i(t)$ flows through the resistor and a current $j(t)$ flows through the inductor. There is therefore a current $i(t) + j(t)$ acting to increase the charge $q(t)$ on the capacitor. The equations linking the unknowns $i(t)$, $j(t)$ and $q(t)$ are

$$-iR = -L\frac{\mathrm{d}j}{\mathrm{d}t} = \frac{q}{C} + V(t)$$

and

$$\frac{\mathrm{d}q}{\mathrm{d}t} = i + j\,.$$

Show that q satisfies the equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = -\frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{1}{L}V\,.$$

Consider the problem where the capacitor carries charge Q and the system is kept in a steady state by maintaining $V(t) = -Q/C$. $V(t)$ is then set to zero at $t = 0$. Consider the three cases (a) $L = 8R^2C$, (b) $L = 4R^2C$ and (c) $L = 2R^2C$. Solve for the subsequent evolution in each case. What are the important differences between the three cases? Note that $V(t) = 0$ for $t > 0$ and that appropriate initial conditions at (or just after) $t = 0$ are that $q = Q$ and $dq/dt = -Q/RC$.

11. Find the complementary function, and a particular integral, of the equation

$$y'' - (2+c)y' + (1+c)y = \mathrm{e}^{(1+2c)x} \qquad (\dagger)$$

in the case $c \neq 0$.

(a) Show that there is a solution of the form $y = f(x, c)$, where

$$f(x, c) = A\,\mathrm{e}^x + B\frac{\mathrm{e}^x}{c}(\mathrm{e}^{cx} - 1) + \frac{\mathrm{e}^x}{2c^2}(\mathrm{e}^{2cx} - 2\,\mathrm{e}^{cx} + 1)\,,$$

for any A, B and c ($c \neq 0$).

(b) Find the limit of $f(x, c)$ as $c \to 0$. Hence find the complementary function and a particular integral for ($\dagger$) in the case $c = 0$.

12. Show that the equation

$$-\frac{\eta}{2}\frac{\mathrm{d}C}{\mathrm{d}\eta} = \frac{\mathrm{d}^2C}{\mathrm{d}\eta^2}$$

can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\ln\left(\frac{\mathrm{d}C}{\mathrm{d}\eta}\right) = -\frac{\eta}{2}\,.$$

Hence show that if $\frac{\mathrm{d}C}{\mathrm{d}\eta}(0) = A$ and $C(0) = B$ then

$$C(\eta) = B + A\int_0^\eta \exp(-t^2/4)\,\mathrm{d}t\,.$$

13. (a) Solve the pair of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = ay + bx\,,$$

where a and b are constants, subject to the initial conditions $x(0) = 2$, $y(0) = 1$.

(b) Show that the pair of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - xy\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -y + xy$$

can be transformed into a single first-order equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)\,.$$

Hence, or otherwise, show that

$$\frac{\mathrm{e}^x}{x}\cdot\frac{\mathrm{e}^y}{y}$$

is independent of t.