Mathematical Methods III (B Course): Example Sheet 2 Easter 2021

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1. A function $\psi(x, y)$ is defined in the strip $x \geq 0,\ 0 \leq y \leq a$ and satisfies Laplace's equation, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0\,.$$

If $\psi(x, y) \to 0$ as $x \to \infty$, and satisfies the boundary conditions

$$\psi(x, 0) = \psi(x, a) = 0, \quad \psi(0, y) = \sin(\pi y/a) + 2\sin(2\pi y/a),$$

use the method of separation of variables, or otherwise, to find the appropriate solution of Laplace's equation.

2. The steady temperature $\phi(x, y)$ in a 2-D rectangular enclosure $0 \leq x \leq \pi,\ 0 \leq y \leq b$ satisfies Laplace's equation. The sides $x = 0$, $x = \pi$ and $y = 0$ are all held at a temperature $\phi = 0$, while the side $y = b$ is held at temperature $\phi(x, b) = x\,(\pi - x)$.

Use the method of separation of variables to determine $\phi(x, y)$ in the interior of the rectangle, and determine a series whose sum is equal to the heat flux through the side $y = b$, namely

$$-\int_0^\pi k\, \frac{\partial \phi}{\partial y}(x, b)\; dx\,.$$

3. The transverse displacement $y(x, t)$ of a string obeys the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\, \frac{\partial^2 y}{\partial t^2}\,,$$

where x and t denote position and time respectively and c is a constant. Show that a possible solution is given by

$$y(x, t) = f(x - ct) + g(x + ct)\,,$$

where f and g are any twice-differentiable functions.

Suppose that the string stretches to infinity in both directions and at time $t = 0$ is released from rest, with displacement

$$y(x, 0) = \frac{1}{1 + x^2}\,.$$

Show that the initial conditions can be satisfied by taking $f = g$, for suitable f. Find the solution $y(x, t)$ and sketch its behaviour.

4. A finite string of length L is stopped at $x = 0$ and $x = L$. The transverse displacement of the string obeys the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\, \frac{\partial^2 y}{\partial t^2}\,,$$

where x and t denote position and time respectively and c is a constant.

At $t = 0$ the string is hit with a hammer of width $L/2$ such that the boundary conditions are

$$\begin{aligned} y(x,0) &= 0 \\ \frac{\partial y}{\partial t}(x,0) &= \begin{cases} v & L/4 < x < 3L/4 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Find the transverse displacement $y(x,t)$ for $t > 0$.

5. The temperature $\Theta(x,t)$ of a finite bar $0 < x < l$ obeys the heat conduction equation

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \Theta}{\partial t}\,.$$

Suppose that the ends of the bar are held at zero temperature, i.e. that $\Theta(0,t) = \Theta(l,t) = 0$. Now suppose that the initial temperature distribution $\Theta(x,0)$ at $t = 0$ may be represented by the Fourier sine series

$$\Theta(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)\,,$$

where b_n are constants. Show that the subsequent temperature $\Theta(x,t)$ is given by

$$\Theta(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)\, \exp(-n^2\pi^2\kappa t/l^2)\,.$$

6. A large number N of drunkards emerge from the popular drinking establishment 'Spoonies' at closing time and diffuse throughout Cambridge. Given that the number density $u(r,t)$ of drunks, at distance r from Spoonies and time t after closing time, obeys the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\kappa}\frac{\partial u}{\partial t}\,,$$

where κ is a constant, verify that a suitable solution is

$$u(r,t) = \frac{A}{t}\,\exp\left(\frac{-r^2}{4\kappa t}\right)\,,$$

with A a constant. Show that $N = 4\pi A\kappa$, and show further that the number density of drunks at distance R from Spoonies never exceeds $N/(R^2\pi e)$.