Natural Sciences Tripos
Part IA Mathematics - Course A

Mathematical Methods I

Examples Sheet 2

Henrik Latter*

Michaelmas Term 2021

This sheet provides exercises for the second half of the Michaelmas Term. **ALL** questions should be attempted by students attending Course A lectures. Answers for the basic skills questions are provided at the back of this sheet

**Comments and corrections to hl278@cam.ac.uk*

H. Calculus Revision

Basic Skills

1. Calculate the following derivatives:

   (a) $\frac{d}{dx}(x^2+3)$ (b) $\frac{d^2}{dx^2}(x^4+2x+6)$

   (c) $\frac{d}{dx}(ax^n+bx+c)$ (d) $\frac{d}{dx}(e^x)$

   (e) $\frac{d}{dt}(at+bt^2\sin\theta)$ (f) $\frac{d}{dx}(1+x^{2/3})$

   (g) $\frac{d}{dy}\left(\frac{1}{1+y}\right)$ (h) $\frac{d}{dx}(\ln(x))$

2. By writing the trigonometric functions in terms of exponential functions evaluate the following derivatives:

   (a) $\frac{d}{dx}(\sin x)$ (b) $\frac{d}{d\theta}(\cos\theta)$

   (c) $\frac{d}{dt}(\tan t)$ (d) $\frac{d}{d\omega}(\sin(-i\omega t))$

   (e) $\frac{d}{dx}(\sinh x)$ (f) $\frac{d}{dx}(\cosh x)$

   (g) $\frac{d}{dx}(\tanh x)$ (h) $\frac{d}{dx}(\tanh(2x))$

3. Calculate the following indefinite integrals:

   (a) $\int x^2 dx$ (b) $\int (ax^n+bx+c)dx$

   (c) $\int e^{2x}dx$ (d) $\int 1/x\ dx$

   (e) $\int \sin(y)dy$ (f) $\int x\cos x dx$

   (g) $\int a\sec^2 x dx$ (h) $\int (2\cos^2 x-1)dx$

4. Calculate the following definite integrals:

   (a) $\int_0^3 (x^2+4)dx$ (b) $\int_0^2 (x-a)^2 dx$

   (c) $\int_0^\pi e^{i\theta}d\theta$ (d) $\int_0^\pi \cos(x)dx$

   (e) $\int_{\pi/4}^{-\pi/4} \text{sech}^2 x dx$ (f) $\int_{-\pi/2}^{\pi/2} \sin(2x)dx$

5. Express the following in terms of partial fractions:

(a) $\dfrac{1}{1-x^2}$

(b) $\dfrac{3x}{2x^2+x-1}$

(c) $\dfrac{2(1-x^2)}{1+x-x^2-x^3}$

(d) $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$

Main Questions

6. Calculate the following derivatives:

   (a) $\dfrac{d}{dx}(x\sin x)$ (b) $\dfrac{d}{d\theta}\left(\dfrac{2\theta}{\cos\theta}\right)$

   (c) $\dfrac{d}{dt}(t^2\ln t)$ (d) $\dfrac{d}{dy}(e^y\cos y)$

   (e) $\dfrac{d}{dx}(\cosh x\sinh x)$ (f) $\dfrac{d}{dx}\left(e^{(x^2+2)}\right)$

7. Find $\dfrac{dy}{dx}$ if $y+e^y\sin y=\dfrac{1}{x}$

   (a) by expressing x as a function of y and then finding $\dfrac{dx}{dy}$.

   (b) by differentiating each term with respect to x.

   (c) Show that (a) and (b) give the same answer.

8. By identifying stationary points, asymptotes and intersections sketch the following curves:

   (a) $y=(x-3)^3+2x$ (b) $y=\dfrac{x}{1+x^2}$

   (c) $y=xe^x$ (d) $y=\dfrac{\ln x}{x+1}$

   (e) $y=\dfrac{1}{1-e^x}$ (f) $y=e^x\cos x$.

9. Evaluate the following definite integral

   $$y=\int_1^2\frac{3}{2x^2+x-1}dx.$$

10. Evaluate the following indefinite integrals:

    (a) $\int -\sin x\cos^5 x\ dx$ (b) $\displaystyle\int\frac{\sec^2 x}{\tan x}\ dx$

    (c) $\displaystyle\int\frac{-2}{x^2-1}\ dx$ (d) $\displaystyle\int\frac{3x^2+2}{x^3+2x-1}\ dx$

    (e) $\int x(1+2x)^{-3/2}\ dx$ (f) $\int\sin x\sec x\ dx$

    (g) $\int\cos^4 x\ dx$ (h) $\int\sin^3 x\ dx$

11. With the help of suitable substitutions, find the following indefinite integrals:

    (a) $\displaystyle\int \tan\theta\sqrt{\sec\theta}\ d\theta$

    (b) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\ dx$

    (c) $\displaystyle\int \frac{8-2x}{(6x-x^2)^{1/2}}\ dx$

    (d) $\displaystyle\int \frac{1}{\sin x}\ dx$

    (e) $\int \sec x \tan x\ dx$

12. Using integration by parts, or otherwise, find the indefinite integrals:

    (a) $\displaystyle\int \ln(x^3)\ dx$

    (b) $\displaystyle\int (\ln x)^3\ dx$

13. Use integration by parts to evaluate the following:

    (a) $\displaystyle\int_0^y x^2 \sin x\ dx$

    (b) $\displaystyle\int_1^y x \ln x\ dx$

    (c) $\displaystyle\int_0^y \sin^{-1} x\ dx$

    (d) $\displaystyle\int_1^y \frac{\ln(a^2+x^2)}{x^2}\ dx$

14. Let $I = \displaystyle\int_0^x e^{at}\cos bt\ dt$ and $J = \displaystyle\int_0^x e^{at}\sin bt\ dt$, where a and b are constants. Use integration by parts to:

    (a) Show that $I = b^{-1}e^{ax}\sin bx - ab^{-1}J$.

    (b) Find another similar relationship between I and J

    (c) Hence find I and J.

    (d) By considering the integral $I + iJ$, find I, J using a different method.

15. Using integration by parts, find a relationship between suitably defined I_n and I_{n-1} or I_{n-2}, where n is any positive integer and hence evaluate the integrals:

    (a) $I_4 = \displaystyle\int_0^1 (1-x^3)^4\ dx$

    (b) $I_6 = \displaystyle\int_0^{\pi/2} x^6 \sin x\ dx$

(c) $I_5 = \int_0^1 x^5 e^x \, dx.$

16. $J(m, n)$, for non-negative integers m and n, is defined by the integral

$$J(m, n) = \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$$

(a) Evaluate

(i) $J(0, 0)$ (ii) $J(0, 1)$,
(iii) $J(1, 0)$ (iv) $J(1, 1)$,
(v) $J(m, 1)$ (vi) $J(1, n)$

(b) Using integration by parts, prove that, for m and n both > 1.

$$J(m, n) = \frac{m-1}{m+n} J(m-2, n) \text{ and } J(m, n) = \frac{n-1}{m+n} J(m, n-2),$$

(c) Evaluate

(i) $J(5, 3)$ (ii) $J(6, 5)$ (iii) $J(4, 8)$.

17. State whether the following functions are even, odd or neither:

(a) x (b) $(x-a)^2$
(c) $\sin x$ (d) $\sin(\pi/2 - x)$
(e) $\exp x$ (f) $|x| \cos x$
(g) $\sqrt{x}$ (h) 2
(i) $\ln \left| \frac{(1+x)}{(1-x)} \right|$

I. Power Series

1. (a) Find the first four terms in the Taylor expansion of $\sin x$ about $x = \pi/6$.

   (b) Hence find an approximate value for $\sin 31°$.

2. Find the first 2 non-zero terms in the Maclaurin Series of the following.

   (a) $\cos x$

   (b) $\sin^{-1} x$

   (c) e^x

   (d) $\ln(x + 1)$

3. (a) Derive an expression for the binomial expansion of $(1 + x)^n$ near $x = 0$ in terms of a sum from 0 to n.

   (b) Hence calculate an approximate expression for $(1 + x)^8$ around $x = 0$.

4. Calculate the first three non-zero terms in the binomial approximation for the following:

   (a) $(1 + x)^{3/2}$

   (b) $(4 + 3x)^{1/2}$

   (c) $(3 - x)^{-1}$

5. Show that for $|x| \leq 1$

$$\tan^{-1} x = x - x^3/3 + x^5/5 - ...$$

   (a) Using the result $\tan^{-1}(1) = \pi/4$, how many terms of the series are needed to calculate π to 10 decimal places?

   (b) Show that $\pi/4 = \tan^{-1}(1/2) + \tan^{-1}(1/3)$ and deduce another series for π

6. (a) Write down the Taylor series for a function f evaluated at $x + h$ in terms of $f(x)$ and its derivatives evaluated at x. Use this result to show that if x_0 is an approximate solution of the equation $f(x) = 0$, then a better approximation is given, in general, by $x = x_1$ where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} .$$

   (b) Sketch the graph of $g(x) = x^3 - 3x^2 + 2$.

   Use the above formula with an initial guess $x_0 = 2.5$ to obtain an improved estimate, x_1 of the largest root of the equation $g(x) = 0$. Apply the method a second time to get a further estimate x_2 (you may leave your answer in rational form).

   Provide a sketch showing the progress of the iterations and demonstrate that the sequence x_0, x_1, x_2, x_3 ... will converge to the root.

J. Probability

1. Two balls are drawn (without replacement) from a box containing 5 blue, 4 green and 1 yellow ball. Like-coloured balls cannot be distinguished.

   (a) Describe the sample space of unordered outcomes.

   (b) Calculate the probability of each outcome.

2. A box of 100 gaskets contains 10 with type A defects only, 5 with type B defects only and 2 with both types of defect.

   Given that a gasket drawn at random has a type A defect find the probability that it also has a type B defect.

3. Show that if there are 23 people in a room, the probability that no two of them share the same birthday is less than 50%.

4. You and a colleague are playing in a gameshow where you both have to select one box each from nine, apparently identical boxes. Six contain a valuable prize but the other three are empty. The host makes you both choose a separate box in turn.

   (a) If you choose first, what is the probability that you win a prize?

   (b) If you choose first and win a prize, what is the probability that your colleague also wins a prize?

   (c) If you choose first and do not win a prize, what is the probability that your colleague does win a prize?

   (d) Is it in your better interest to persuade your colleague to choose first?

5. You randomly choose a biscuit from one of two identical jars. Jar A has 10 chocolate covered biscuits and 30 plain ones. Jar B has 20 chocolate covered and 20 plain biscuits. Unfortunately you choose a plain biscuit. What is the probability that you chose from Jar A?

6. A weighted die gives a probability p of throwing 2,3,4, or 5 , probability $2p$ of throwing 6 and probability $p/2$ of throwing 1.

   (a) Calculate p

   (b) Calculate $\langle x \rangle$, the expected mean score after many throws of the dice.

   (c) In a single throw, what is the probability of obtaining a score higher than $\langle x \rangle$?

   (d) Calculate the variance, σ^2

   (e) Check the formula, $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$.

7. Two duellists, A and B, take alternate shots at each other, and the duel is over when a shot (fatal or otherwise!) hits its target.

   Each shot fired by A has a probability, α, of hitting B

   Each shot fired by B has a probability, β, of hitting A

   (a) Calculate the probability P_1 that A will win if he fires the first shot

   (b) Calculate the probability P_2 that A will win if B fires the first shot

   (c) If they agree to fire at the same time, rather than alternatively, what is the probability P_3 that A will win (i.e. hit B without being hit himself).

8. In the National Lottery, 6 balls are drawn at random from 49 balls, numbered from 1 to 49. You pick 6 different numbers.

   (a) What is the probability that your 6 numbers match those drawn?

   (b) What is the probability that exactly r of the numbers you choose match those drawn?

   (c) What is the probability that 5 numbers you choose match those drawn and your 6th number matches a bonus ball drawn after the other 6?

K. Probability Distributions

1. You arrive home after a big night out and attempt to open your front door with one of the three keys in your pocket. (You may assume that exactly one key will open the door and that if you use it you will be successful).
   Let X, a random variable, be the number of tries that you need to open the door if you take the keys at random from your pocket, but drop any key that fails onto the ground.
   Let Y, another random variable, be the number of tries needed if you take the keys at random from your pocket and immediately put back into your pocket any key that fails.

   Find the probability distribution for X and Y and evaluate $\langle X\rangle$ and $\langle Y\rangle$.

   (*Hint: it may be useful to note that* $1+2x+3x^2+...=(1-x)^{-2}$ *if* $|x|<1$)

2. An opaque bag contains 10 green counters and 20 red. One counter is selected at random and then replaced: green scores 1 and red scores zero. Five draws are made.

   (a) Calculate p_r, the probability of obtaining score $r=\{0,1,2,3,4,5\}$. Check that the probabilities sum to 1. Write down the mean $\langle r\rangle$ and variance σ^2 of the score.

   (b) Calculate the probability of obtaining scores in the ranges $\langle r\rangle\pm\sigma/2$, $\langle r\rangle\pm\sigma$.

   (c) The Gaussian approximation of the binomial distribution in (a) is given as: $P_1(r)\propto\exp[-9(r-\frac{5}{3})^2/20]$
   Sketch $P_1(r)$, and p_r.

   (d) Compare your answers in (b) with those for $P_1(r)$. In what sense is $P_1(r)$ a good approximation to p_r?

   (e) Which of your answers would have been different had you not replaced the counters after each selection?

   (f) Which of your answers would have been different had the bag contained only 1 green counter and 2 red counters?

3. The sizes of raindrops in a storm has a probability density function given by:

$$f(s)=\begin{cases}10ds^2, & 0\le s\le 0.6,\\ 9d(1-s), & 0.6<s\le 1,\\ 0, & s>1,\end{cases}$$

   where d is a constant.

   (a) Find the value of d and sketch the graph of this distribution.

   (b) Write down the most likely size of a raindrop.

   (c) Find the mean size of the raindrops.

   (d) Determine the probability that the size will be: (i) more than 0.8; (ii) between 0.4 and 0.8.

4. The lifetime of a bulb in a traffic signal is a random variable with density:

$$f(t) = \begin{cases} 1, & 1 \leq t \leq 2, \\ 0, & \text{otherwise}, \end{cases}$$

where t, is measured in years.

(a) What is the probability as a function of y that the failure time of a bulb is less that y years?

(b) The traffic signal contains 3 bulbs. What is the probability as a function of z that none of the bulbs have to be replaced in z years?

5. A certain disease is known to afflict one in a thousand people. You take a medical test that is said to be 99% accurate (i.e. it gives you the correct result in 99% of the cases in which it is used).

What is the probability that you actually have the disease if the test says you do? Discuss the assumptions implicit in this question and your answer.

6. Suppose that n distinguishable particles are placed randomly into N boxes (states). A particular configuration of this system is such that there are n_s particles in state s, where $1 \leq s \leq N$. The ordering of particles in any particular state does not matter. Show that the number of ways of realising a particular configuration is:

$$W = n! \prod_{s=1}^{N} \frac{1}{n_s!}$$

[NB: The product symbol $\prod$ is defined such that $\displaystyle\prod_{s=1}^{N} a_s = a_1 a_2 ... a_N$].

Numerical Answers to Basic Skills

H. Calculus Revision

1. (a) $2x$ (b) $12x^2$ (c) $anx^{n-1} + b$
(d) e^x (e) $a + 2bt\sin\theta$ (f) $\frac{2}{3}x^{-1/3}$
(g) $\frac{-1}{(1+y)^2}$ (h) $\frac{1}{x}$

2. (a) $\cos x$ (b) $-\sin\theta$ (c) $\sec^2 t$
(d) $-it\cosh(wt)$ (e) $\cosh x$ (f) $\sinh x$
(g) $\text{sech}^2 x$ (h) $2\text{sech}^2 2x$

3. (a) $\frac{x^3}{3} + const$ (b) $\frac{a}{n+1}x^{n+1} + \frac{b}{2}x^2 + cx + const$
(c) $\frac{1}{2}e^{2x} + const$ (d) $\ln x + const$
(e) $-\cos y + const$ (f) $x\sin x + \cos x + const$
(g) $a\tan x + const$ (h) $\frac{1}{2}\sin 2x + const$

4. (a) 21 (b) $\frac{8-12a+6a^2}{3}$ (c) 2i
(d) 0 (e) $-2\tanh(\pi/4) = -1.312$ (f) 0

5. (a) $\frac{1}{2(1-x)} + \frac{1}{2(1+x)}$
(b) $\frac{1}{2x-1} + \frac{1}{1+x}$
(c) $\frac{2}{(1+x)}$
(d) $\frac{2x+3}{(x^2+2x+3)} + \frac{1}{1+x} + \frac{x}{1-x}$