Example Sheet I
Mathematical Methods III:
Course A Easter Term 2022

Natural Sciences 1A, Computer Science 1A

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This sheet provides exercises covering the material contained in the first **7** lectures of the Easter Term. **ALL** questions should be attempted by students attending Course A lectures.

Students should attempt questions as soon as the work has been covered in lectures - do not wait for your supervisors to set questions. In the first lecture there are some quick quiz questions which can be used for discussion in the first supervision.

These questions can be checked using Mathematica or Matlab and students are strongly encouraged to do so. Questions should be done manually first and then checked using Mathematica or Matlab.

1 Matrix Theory

1. Write the system of equations

$$\begin{aligned} \pi\, p + 2z &= 2 \\ 2\, p + y + \sqrt{3}\, w &= 4 \\ y + 3w &= 3 \\ a^2\, z + p + 3\, y &= 6 \end{aligned}$$

   where a is a real parameter, in a matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$ in terms of the column vectors

$$\mathbf{x} = \begin{pmatrix} p \\ y \\ z \\ w \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}$$

   but do not bother to solve the equations.

2. (a) Show that the line $\mathbf{x} = \mathbf{a} + \alpha\, \mathbf{t}$, where α is a real parameter, can be written as

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

   in terms of the components $\mathbf{x} = (x, y, z)$, $\mathbf{a} = (a, b, c)$, $\mathbf{t} = (l, m, n)$ of the vectors $\mathbf{x}$, $\mathbf{a}$, $\mathbf{t}$.

   (b) Write a set of equations which determine the coordinates of the point where this line intersects the plane

$$p\, x + q\, y + r\, z = s\,,$$

   in matrix form $\mathbf{A}\mathbf{x} = \mathbf{d}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

3. Solve the system of equations

$$\begin{pmatrix} 6 & 7 & 3 & -1 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -12 \\ 9 \\ 2 \\ 7 \end{pmatrix},$$

   for x, y, z and w.

4. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -3 \\ -1 & 0 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

   State which of the following products do not exist (*i.e.* are inconsistent with the rules of matrix multiplication) and evaluate those that do

   (a) $\mathbf{A}^2$, (b) $\mathbf{AB}$, (c) $\mathbf{AC}$, (d) $\mathbf{CA}$, (e) $\mathbf{B}^2$, (f) $\mathbf{BC}$, (g) $\mathbf{CB}$ and (h) $\mathbf{C}^2$.

5. Consider the column matrix $\mathbf{u}$ and row matrix $\mathbf{v}$:

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 & 2 & b \end{pmatrix}.$$

   (a) Show that $\mathbf{vu} = 2 + 2\,b$.
   (b) Evaluate the 3×3 matrix $\mathbf{uv}$.

6. Solve the following equations

$$3\,x - 2\,y = 9\,, \qquad x + 3y = 3$$

   by the method of Gaussian elimination.

7. Use the method of Gaussian elimination to investigate the following linear equations

$$\begin{aligned} x + y + a\,z &= 1 \\ 3x + 4y + (2 + 3\,a)z &= 5 \\ y - x + z &= b \end{aligned}$$

   You should find the values of a and b which give rise to:

   (a) unique solutions
   (b) no solutions
   (c) many solutions

8. Identify the 3×3 symmetric matrix $\mathbf{A}$ such that the equation

$$x^2 + y^2 + 9\,z^2 + 4x\,y + 6x\,z = 1$$

   can be written in the matrix form

$$\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\,.$$

9. Let

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

   (a) Compute the following quantities:

   (i) $\mathbf{M}^2$ (ii) $\mathbf{M}\,\mathbf{M}^T$ (iii) $\mathbf{M}^T\mathbf{M}$.

   (b) Express $\mathbf{M}$ as a sum of a symmetric and antisymmetric matrix.
   (c) Let $\mathbf{D}$ be an $(N \times M)$ matrix. For what values of N and M do

   (i) $\mathbf{D}\,\mathbf{D}^T$ and (ii) $\mathbf{D}^T\mathbf{D}$

   exist?

10. Show, by producing specific examples (2×2 matrices should suffice), that:

    (a) the product of two non-zero matrices can be zero;

    (b) $\mathbf{AB}$ can be zero even if $\mathbf{BA}$ is non-zero;

    (c) the product of two non-zero symmetric matrices can be anti-symmetric.

11. The matrices $\mathbf{A}$ and $\mathbf{B}$ and the vector $\mathbf{x}$ have elements a_{ij}, b_{ij} and x_i.

    (a) What are the elements of $\mathbf{A}^T$?

    (b) Write down the matrices represented by

    (i)
$$\sum_{j,k=1}^{3} a_{ij}\, b_{jk}\, x_k$$

    (ii)
$$\sum_{j=1}^{3} x_j\, b_{ij}$$

    (iii)
$$\sum_{i,j=1}^{3} a_{ij}\, b_{kj}\, x_i$$

    (iv)
$$\sum_{j,k=1}^{3} a_{ij}\, a_{kj}\, a_{km}$$

12. If
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix},$$

    calculate:

    (a) $\det \mathbf{A}$, (b) $\det \mathbf{B}$, (c) $\det(\mathbf{AB})$ and (d) $\det(\mathbf{A}^{-1})$.

13. Calculate the determinant of the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

    using the following procedures

    (a) by row operations (Gaussian elimination);

    (b) by expanding along the first row (*i.e.* by calculating $a_{11}\,\Delta_{11} + a_{12}\Delta_{12} + a_{13}\Delta_{13}$)

    (c) by expanding down the first column (*i.e.* by calculating $a_{11}\,\Delta_{11} + a_{21}\,\Delta_{21} + a_{31}\,\Delta_{31}$)

    where Δ_{ij} is the cofactor of a_{ij} in $\mathbf{A}$.

14. Find the inverse of
$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix},$$

    and hence solve for $\mathbf{x}$ in
$$\mathbf{Ax} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

15. Cramer's rule for writing down the solution of the linear equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

is

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}},$$

where $\mathbf{A}_i$ is the matrix obtained from $\mathbf{A}$ by replacing the $i-$th column by the column vector $\mathbf{b}$. Using Cramer's rule, solve for the system of equations of problem number 7.

16. Find the rotation matrix $\mathbf{R}$ such that $\mathbf{y} = \mathbf{R}\mathbf{x}$ represents an anticlockwise rotation of the axes about the $z-$axis through an angle of $\pi/2$ radians.

17. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{pmatrix}.$$

18. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{M} = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

then verify that

(a) the eigenvalues are real

(b) the sum of the eigenvalues is equal to $\operatorname{tr}\mathbf{M}$

(c) the product of the eigenvalues is equal to $\det \mathbf{M}$

(d) the eigenvectors are orthogonal

19. Verify that the following matrix is orthogonal:

$$\mathbf{Q} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

(a) Given that it represents a rotation, what can be said about one of its eigenvalues?

(b) Find the axis of rotation.

(c) By considering the effect of the matrix on a vector orthogonal to the axis of rotation find the angle of rotation.

20. Write the conic

$$x^2 + 16\,x\,y - 11\,y^2 = 1$$

in the form

$$\mathbf{x}^T \mathbf{A}\,\mathbf{x} = 1,$$

where $\mathbf{A}$ is a symmetric 2×2 matrix.

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Write down the equation of the conic referred to new axis along the directions of the eigenvectors.