4 Differentiation

Rates of change, usually with respect to time and space, underpin so many of our scientific theories of the world. They regularly appear in the governing *differential* equations of a theory. There are almost too many examples to list, but prominent equations include: Newton's second law (dynamics), Schroedinger's equation (quantum mechanics), Einstein's field equations (general relativity), the Navier-Stokes equation (fluid dynamics), the Malthus and logistic models (population growth), Fisher's equation (gene propagation), and chemical reaction kinetics (chemistry).

In this part of the course we revise the basics of *differentiation*, which provides the mathematical foundations of change. We focus only on functions of a single variable.

4.1 First Principles

The derivative of a function $y(x)$ at a given point x will be denoted dy/dx and is defined by the limiting process:

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}\ .} \tag{151}$$

Geometrically, the derivative is the gradient of the tangent line to the curve given by $y(x)$ at the point x. The tangent line has a slope such that it only just touches the curve at this point.

Example: differentiate $y = x^3$ from first principles:

$$\begin{aligned} y(x+\delta x) &= (x+\delta x)^3 = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \\ y(x+\delta x) - y(x) &= 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3\ . \end{aligned} \tag{152}$$

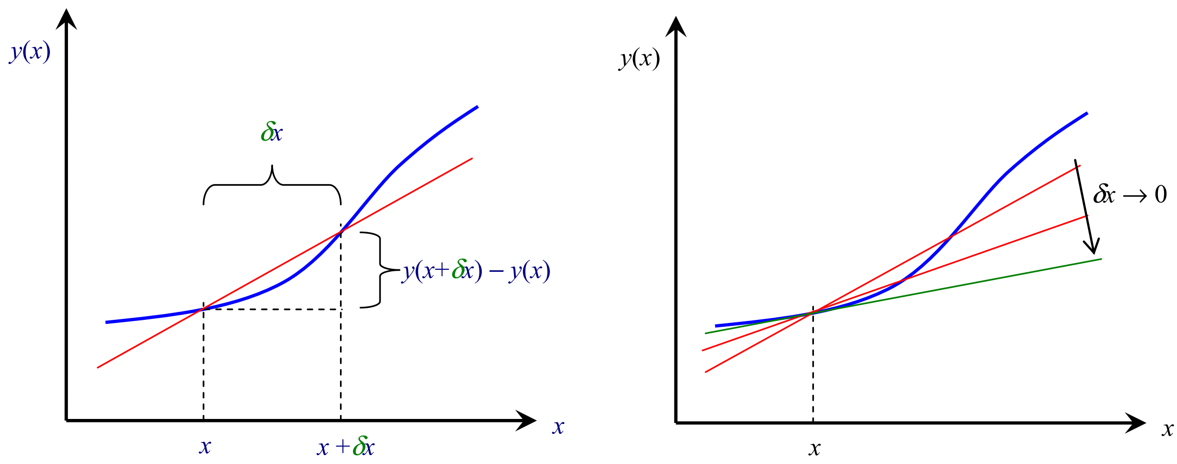

Figure 35: Definition of differentiation as the gradient of a curve.

Now

$$\frac{y(x+\delta x)-y(x)}{\delta x} = 3x^2 + 3x(\delta x) + (\delta x)^2 \; .$$

Take the limit $\delta x \to 0$ and the second and third terms on the right disappear, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} \equiv \lim_{\delta x \to 0} \frac{y(x+\delta x)-y(x)}{\delta x} = 3x^2 \; . \qquad (153)$$

4.1.1 Differentiability

Functions are not necessarily differentiable everywhere.

Example 1: consider the *Heaviside step function* $\mathrm{H}(x)$, defined so that $\mathrm{H}(x) = 0$ for $x < 0$ and $\mathrm{H}(x) = 1$ for $x \geq 0$. What is its derivative at $x = 0$?

If we approach the limit in the definition of the derivative from negative values of δx, then $[y(0) - y(\delta x)]/\delta x = 1/\delta x$, which diverges as $\delta x \to 0$. So the derivative at $x = 0$ using (151) cannot exist.

This is an example of a *discontinuous function*; such functions are not differentiable at their discontinuities.

Example 2: consider the absolute value function $y(x) = |x|$, which is continuous but *not smooth* at $x = 0$.

At $x = 0$, using the formal definition of the derivative, we obtain $d|x|/dx = 1$ if we approach the limit from above $x = 0$ (positive δx), and $d|x|/dx = -1$ if we approach the limit from below $x = 0$ (negative δx). We conclude that the derivative is not well-defined, as it depends on which direction you take the limit.

For a function $y(x)$ to be differentiable at a point x, the function must be both **continuous** and **smooth**.

4.1.2 Higher order derivatives

The derivative dy/dx is a function of x, so we can differentiate it again (assuming it is smooth and continuous). This is the *second derivative*, which is denoted by

$$\frac{d^2y}{dx^2} \equiv \frac{d}{dx}\left(\frac{dy}{dx}\right) , \tag{154}$$

It measures the rate of change of the slope, i.e. its *curvature*.

The notation for going further and taking the nth derivative is

$$\frac{d^ny}{dx^n} \equiv \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) . \tag{155}$$

So, for the example with $y = x^3$, we have

$$\begin{aligned} \frac{\mathrm{d}x^3}{\mathrm{d}x} &= 3x^2\,, \\ \frac{\mathrm{d}^2(x^3)}{\mathrm{d}x^2} &= \frac{\mathrm{d}(3x^2)}{\mathrm{d}x} = 6x\,, \\ \frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} &= \frac{\mathrm{d}(6x)}{\mathrm{d}x} = 6\,, \\ \frac{\mathrm{d}^4(x^3)}{\mathrm{d}x^4} &= \frac{\mathrm{d}(6)}{\mathrm{d}x} = 0\,, \end{aligned} \tag{156}$$

where all derivatives of higher order than the fourth are zero.

4.1.3 Alternative notations

The dy/dx notation for the derivative of $y(x)$ was proposed by Leibniz. However, Newton originally had a more compact notation using dots (or primes):

$$\begin{aligned} \dot{y} &= \frac{dy}{dx} \quad \text{or} \quad y' = \frac{dy}{dx}\,, \\ \ddot{y} &= \frac{d^2y}{dx^2} \quad \text{or} \quad y'' = \frac{d^2y}{dx^2}\,. \end{aligned}$$

For higher order derivatives it can be unwieldy to employ dots and dashes. Generally we use the more compact notation for the n^{th} derivative

$$y^{(n)}(x) = \frac{d^ny}{dx^n}.$$

Note that some people use Roman numerals with this convention, so that $d^4y/dx^4 = y^{\mathsf{iv}}(x)$ and $d^5y/dx^5 = y^{\mathsf{v}}(x)$.

4.2 Derivatives of elementary functions

Little progress is possible in calculus without knowing the basic derivatives of elementary functions, including powers of x, trigonometric, exponential and logarithmic functions.

You should have the following on automatic recall:

$$y = x^n \qquad \Rightarrow \qquad \frac{dy}{dx} = nx^{n-1}\,,$$
$$y = e^x \qquad \Rightarrow \qquad \frac{dy}{dx} = e^x,$$
$$y = \ln x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x}\,,$$
$$y = \sin x \qquad \Rightarrow \qquad \frac{dy}{dx} = \cos x\,,$$
$$y = \cos x \qquad \Rightarrow \qquad \frac{dy}{dx} = -\sin x\,,$$
$$y = \tan x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\cos^2 x}\,.$$

You may not be as familiar with the derivatives of the hyperbolic functions introduced in Section 3:

$$y = \sinh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \cosh x\,,$$
$$y = \cosh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \sinh x,$$
$$y = \tanh x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\cosh^2 x}\,.$$

However, these are easy to derive using the definitions of the hyperbolic functions. For example, differentiating $\sinh x$ we get

$$\frac{d\sinh x}{dx} = \frac{d}{dx}\left[\frac{1}{2}\left(e^x - e^{-x}\right)\right] = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x\,.$$

The hyperbolic and trigonometric cases are similar, but note the sign difference between the derivatives of $\cosh x$ and $\cos x$.

4.3 Rules for differentiation

4.3.1 The product rule

Sometimes we are given a product of functions in the form

$$y(x) = u(x)v(x)\,, \tag{157}$$

where we know how to differentiate the factors $u(x)$ and $v(x)$ individually.

The rule for differentiating this product of functions is the following:

$$\boxed{\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}} \tag{158}$$

This result can be produced from first principles relatively quickly:

$$\begin{aligned}
\frac{y(x+\delta x) - y(x)}{\delta x} &= \frac{u(x+\delta x)v(x+\delta x) - u(x)v(x)}{\delta x} \\
&= \frac{u(x+\delta x)v(x+\delta x) - u(x)v(x+\delta x)}{\delta x} \\
&\qquad + \frac{u(x)v(x+\delta x) - u(x)v(x)}{\delta x} \\
&= \left[\frac{u(x+\delta x) - u(x)}{\delta x}\right] v(x+\delta x) + u(x)\left[\frac{v(x+\delta x) - v(x)}{\delta x}\right] .
\end{aligned}$$

We now take the limit $\delta x \to 0$ and get the result.

Example 4.1 Differentiate $y = \ln x \sin x$.

First set $u = \ln x$ and $v = \sin x$. Then:

$$\frac{d(\ln x \sin x)}{dx} = \frac{d \ln x}{dx}\sin x + \ln x \frac{d \sin x}{dx} = \frac{\sin x}{x} + \ln x \cos x.$$

4.3.2 The chain rule

Often we are given complicated expressions in which we have a function $y = f(u)$ with $u = u(x)$ itself being a function of x (e.g. $y = f(u) = \sin u$ and $u(x) = x^2$, so that $y = \sin x^2$). The method for differentiating a 'function of a function' is called the *chain rule* and is given by

$$\boxed{\frac{\mathrm{d}(f(u(x)))}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}} \tag{159}$$

We can understand why this rule arises by writing

$$\frac{f(u(x+\delta x))-f(u(x))}{\delta x}=\left[\frac{f(u(x+\delta x))-f(u(x))}{u(x+\delta x)-u(x)}\right]\left[\frac{u(x+\delta x)-u(x)}{\delta x}\right]. \tag{160}$$

Next write $\delta u = u(x+\delta x)-u(x)$, i.e. the accompanying small change in the function u due to the small change δx in x.

We then have:

$$\frac{f(u(x+\delta x))-f(u(x))}{\delta x}=\left[\frac{f(u+\delta u)-f(u)}{\delta u}\right]\left[\frac{u(x+\delta x)-u(x)}{\delta x}\right], \tag{161}$$

and we now take the limit $\delta x \to 0$ (so that necessarily $\delta u \to 0$ as well). The first factor becomes df/du and the second factor du/dx.

Example 4.2 Differentiate $\sin x^2$ and $\ln(\cos x)$ with respect to x.

So we can write $f(u)=\sin u$ where $u=x^2$, and we want to find df/dx. First off, we have

$$\frac{df}{du}=\cos u, \qquad \frac{du}{dx}=2x.$$

The using the chain rule we have:

$$\frac{df}{dx}=\frac{df}{du}\frac{du}{dx}=(\cos x^2)2x.$$

Okay, for the second problem, let us set now $f(u)=\ln u$ and $u=\cos x$. We note:

$$\frac{df}{du}=\frac{1}{u}, \qquad \frac{du}{dx}=-\sin x,$$

thus

$$\frac{df}{dx}=\frac{1}{\cos x}(-\sin x)=-\tan x.$$

4.3.3 The quotient rule

We have already seen how to differentiate the product uv, now we consider the quotient u/v. In this case, we can find the derivative from the formula:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v(\mathrm{d}u/\mathrm{d}x) - u(\mathrm{d}v/\mathrm{d}x)}{v^2}} \tag{162}$$

This result comes about via the product and chain rules. First, write the quotient u/v as the product

$$\frac{u}{v} = u \times \left(\frac{1}{v}\right) . \tag{163}$$

Now differentiate (163) using the product rule (158) to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \left(\frac{1}{v}\right) + u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) . \tag{164}$$

Next use the chain rule (159) to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{v}\right) = \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{1}{v}\right)\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} . \tag{165}$$

Finally substitute this result back into (164) to find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} , \tag{166}$$

and group terms over the common denominator v^2.

<u>Example 4.3</u> Differentiate $(\sin x)/x$ with respect to x.

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{xd(\sin x)/dx - \sin x(dx/dx)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

4.3.4 Implicit differentiation

It is also possible to find the derivative of y with respect to x from an equation of the form

$$g(y) = f(x)\,, \tag{167}$$

where g and f are given functions. For example: $\mathrm{e}^y \cos y = x \cos x$. Here, the exact dependence of y on x may not be known explicitly at all. Rather, it is *implicit*.

Using the chain rule:

$$\frac{\mathrm{d}g(y)}{\mathrm{d}x} = \frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}\,, \tag{168}$$

so that differentiating (167) with respect to x we have

$$\frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}\,. \tag{169}$$

Rearranging, we find that

$$\boxed{g(y) = f(x) \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f/\mathrm{d}x}{\mathrm{d}g/\mathrm{d}y}\,.} \tag{170}$$

An important special case gives the 'reciprocal rule'. Suppose we want to differentiate $y(x)$ but only know the derivative of its inverse function $x(y)$, i.e. dx/dy.

Okay, set $f(x) = x$ in (167), so now $x = g(y)$. Then we have immediately

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}}\,. \tag{171}$$

<u>Example 4.4</u> Find the derivative of $y = \tan^{-1} x$ with respect to x.

Let us write $x = \tan y$ and then take the y derivative:

$$\frac{dx}{dy} = \frac{d\tan y}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

But the reciprocal rule says:

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1+x^2},$$

So $d\tan^{-1} x/dx = 1/(1+x^2)$.

Example 4.5 A circle has equation $y^2 = 9 - (x-1)^2$. Find the gradient.

We want to find an expression for dy/dx. The equation for a circle is of the form $g(y) = f(x)$, with $g = y^2$ and $f = 9 - (x-1)^2$. The implicit differentiation law gives us

$$\frac{dy}{dx} = \frac{df/dx}{dg/dy} = \frac{-2(x-1)}{2y} = \pm\frac{(x-1)}{\sqrt{9-(x-1)^2}}.$$

Example 4.6 [2006 paper 2, Question 1A]. If

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

find dy/dx as a function of x.

First write $\sin y = x/\sqrt{1+x^2}$, which is in the form $g(y) = f(x)$ and use the implicit differentiation formula. We have $dg/dy = \cos y$. We also have, using the quotient rule:

$$\begin{aligned}\frac{df}{dx} = \frac{d}{dx}\frac{x}{\sqrt{1+x^2}} &= \frac{\sqrt{1+x^2}(dx/dx) - (d\sqrt{1+x^2}/dx)x}{1+x^2}, \\ &= \frac{\sqrt{1+x^2} - [x(1+x^2)^{-1/2}]x}{1+x^2}, \\ &= \frac{1}{(1+x^2)^{3/2}}.\end{aligned}$$

Putting the two results together:

$$\frac{dy}{dx} = \frac{df/dx}{dg/dy} = \frac{1}{\cos y (1+x^2)^{3/2}}.$$

It would be nice to have the RHS in terms of x only. To do this, note

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{x^2}{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

If we put this into our formula for dy/dx, we get simply:

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

4.4 Stationary points

A stationary point (also called a 'turning point') of the curve $y = f(x)$ is a point where $\mathrm{d}y/\mathrm{d}x = 0$.

Stationary points can be classified using the following rules:

- If $\mathrm{d}^2y/\mathrm{d}x^2 > 0$ at the stationary point, then it is a **minimum**.

  This is because at a minimum the gradient *increases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 < 0$ at the stationary point, then it is a **maximum**.

  This is because at a maximum the gradient *decreases* through the turning point.

- If $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ at the stationary point, then further investigation is required:

1. if the first nonzero derivative $\mathrm{d}^ny/\mathrm{d}x^n \neq 0$ has n odd, then the stationary point is a **point of inflection**;

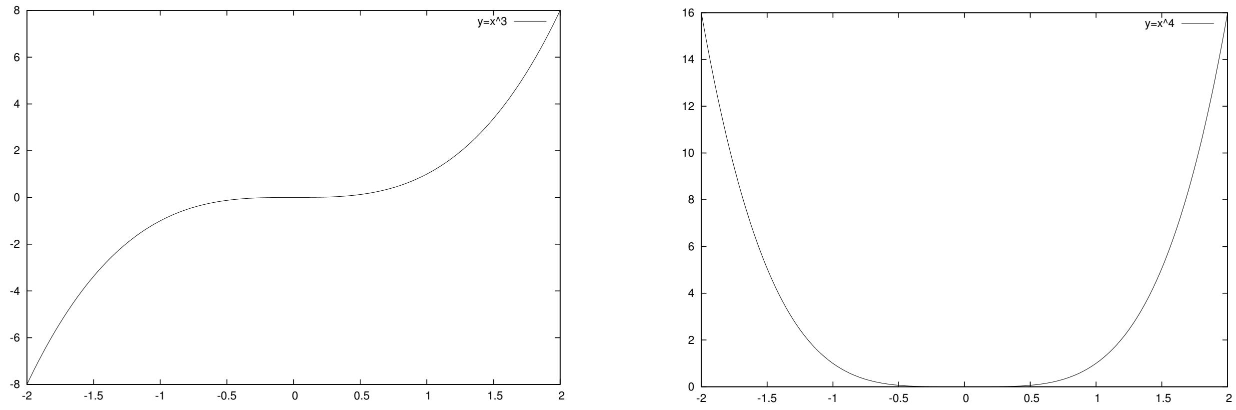

Figure 36: Graphs of $y = x^3$ and $y = x^4$.

2. If the first nonzero derivative $\mathrm{d}^n y/\mathrm{d}x^n \neq 0$ has n even and it is positive, then the stationary point is a **minimum**;
3. If the first nonzero derivative $\mathrm{d}^n y/\mathrm{d}x^n \neq 0$ has n even and it is negative, then the stationary point is a **maximum**.

As examples, consider $y = x^3$ and $y = x^4$. Both have a stationary point at $x = 0$ with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0\,.$$

For $y = x^3$, the point $x = 0$ is a point of inflection because

$$\frac{\mathrm{d}^3(x^3)}{\mathrm{d}x^3} = 6 \neq 0\,. \tag{172}$$

For $y = x^4$, this is a minimum because the first non-zero derivative is even and positive,

$$\frac{\mathrm{d}^4 x^4}{\mathrm{d}x^4} = 24 > 0\,. \tag{173}$$

4.4.1 More on points of inflection

An inflection point is where $d^2y/dx^2 = 0$ and also d^2y/dx^2 changes sign.

- It is where the curve changes from being concave up to concave down or vice versa.
- It need not be a stationary point (i.e. where $dy/dx = 0$).
- If $dy/dx > 0$ at the inflection point it is a 'rising point of inflection'; if $dy/dx < 0$ it is a 'falling point of inflection'.
- Between an adjacent maximum and a minimum there must be a point of inflection.

4.5 Curve sketching

Basic curve sketching techniques are very useful for detemining the *main features* of the overall shape of a function $y = f(x)$. It means we can understand the behaviour of the function without the need to compute it everywhere. In other words, we can get a qualitative idea of what it is about.

When sketching curves there are a number of things to consider:

1. Where does the curve **intercept** the x and y axes – i.e. what is the value of $f(0)$ and what are solutions for $f(x) = 0$?
2. Is there any **symmetry**? Is the function *even*, $f(x) = f(-x)$, or is the function *odd*, $f(x) = -f(-x)$?
3. What are the **asymptotes**? In other words, what is the behaviour as $x \to \pm\infty$ or at any boundaries?
4. Are there any **singularities**, that is, points where the function becomes infinite? These create *vertical asymptotes* about the singular point.
5. What are the **stationary points** (i.e. where does $dy/dx = 0$)? What is their nature – minimum, maximum or point of inflection?

The following examples illustrate important aspects of curve sketching techniques.

Example 4.7 Sketch $y = \exp x - \sin x$.

- Intercepts of the x axis are difficult to explicitly calculate: we need to solve $\exp x = \sin x$. However, graphically we can see that there are an infinite number of solutions, but only for $x < 0$.

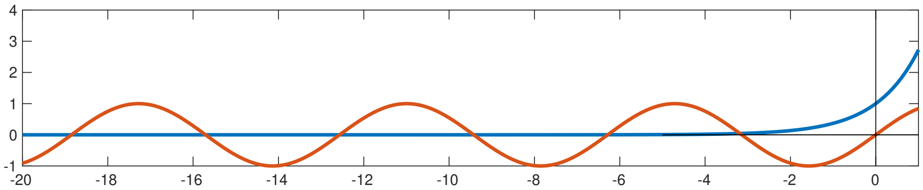

Figure 37: Graphs of $y = e^x$ and $y = \sin x$. The points at which they intercept give us the locations where $y = e^x - \sin x$ crosses the x axis.

  On the other hand, the curve cuts the y axis at $y = \exp(0) - \sin(0) = 1$.

- What about when $x \to \pm\infty$? Algebraically, we can see that as $x \to -\infty$ then y approaches $-\sin x$. But when $x \to \infty$, we see that $y \to e^x$.

- What about stationary points? We see that $dy/dx = e^x - \cos x = 0$ also yields an infinite number of solutions with $x \leq 0$. Graphically we see that there is a turning point at $x = 0$. Is the $x = 0$ turning point a max or min? Well, we have $d^2y/dx^2 = e^x + \sin x$ which is $= 1$ at $x = 0$, and so this turning point is a minimum.

- We probably have enough information to sketch out the curve. Joining up $-\sin x$ and e^x with a minimum at $x = 0$.

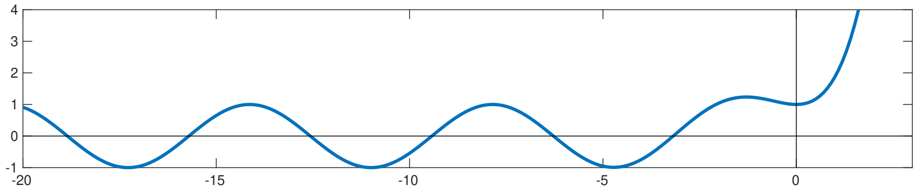

Figure 38: Graph of $y = e^x - \sin x$.

Example 4.8 Sketch $y = \frac{\ln x}{x}$. Show that $e^\pi > \pi^e$.

- The x axis intercept can be gleaned from $y = \ln x/x = 0$, and there is only one: $x = 1$. The curve cannot cross the y axis because of the $\ln$ in the denominator. And, in fact, the function is not even defined when $x < 0$.

- Asymptotic behaviour? Clearly as $x \to 0$, $y \to -\infty$. And when $x \to \infty$ the denominator in y defeats the numerator and $y \to 0$ from above.

- What about stationary points? Differentiating, we have

$$\frac{dy}{dx} = \frac{x(d\ln x/dx) - \ln x(dx/dx)}{x^2} = \frac{1 - \ln x}{x^2}.$$

  We have turning points wherever $dy/dx = 0$, i.e. when $\ln x = 1$ which yields $x = e$, with $y = 1/e$. So just one turning point.

  Is it a max or min? We could look at the second derivative, but that would create more algebra. We do know that $y \to -\infty$ as $x \to 0$, and we know that $y \to 0$ as $x \to \infty$ from above, therefore the turning point inbetween these limits must be a *maximum*.

- We probably have enough to sketch out the curve now:

Let us derive this interesting inequality. We know that $y(x) < y(e)$ for all $x > 0$. So let us consider the point $x = \pi$. We then have $y(\pi) < y(e)$, which becomes

$$\frac{\ln \pi}{\pi} < \frac{\ln e}{e} = \frac{1}{e}.$$

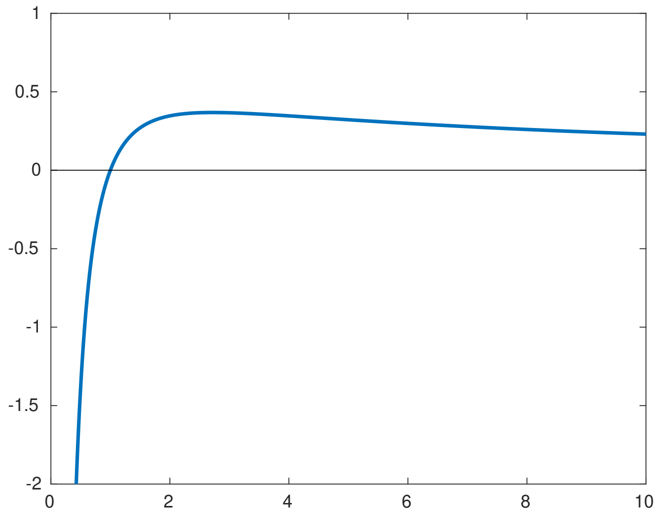

Figure 39: Graph of $y = \ln x/x$.

We rearrange this into $e \ln \pi < \pi$ and then $\ln \pi^e < \pi$. Take the exponential of both sides and we get

$$\pi^e < e^\pi,$$

as desired.

5 Integration

Integration is the *reverse* of differentiation; hence the study of calculus necessitates understanding one and the other.

But quite separately, integration is essential to a myriad of physical processes, involving areas, volumes, averaging, and correlations.

Integral equations (as opposed to differential equations) underpin applications as diverse as population dynamics (Volterra's equation), propagation of fish in a lake, viscoelasticity, electro and magneto-statics, and wave-scattering.

We can think of integration in two ways:

First, we have the *definite integral* over some interval bounded by the points $x = a$ and b:

$$\int_a^b f(x)dx\,. \tag{174}$$

This can be understood as the *area* (a) under the curve $y = f(x)$ and above the x-axis in the xy-plane and (b) bounded between the vertical lines $x = a$ and $x = b$.

Second, we have the *indefinite integral* which is expressed without bounds as

$$\int f(x)dx\,. \tag{175}$$

This can be regarded as the *inverse operation to differentiation* or the 'antiderivative', and it yields another function $F(x)$.

5.1 Integration as area – definite integrals

Consider the curve $y = f(x)$ in the range $a \leqslant x \leqslant b$. We can approximate the area under the curve by dividing the range up into N small subintervals of

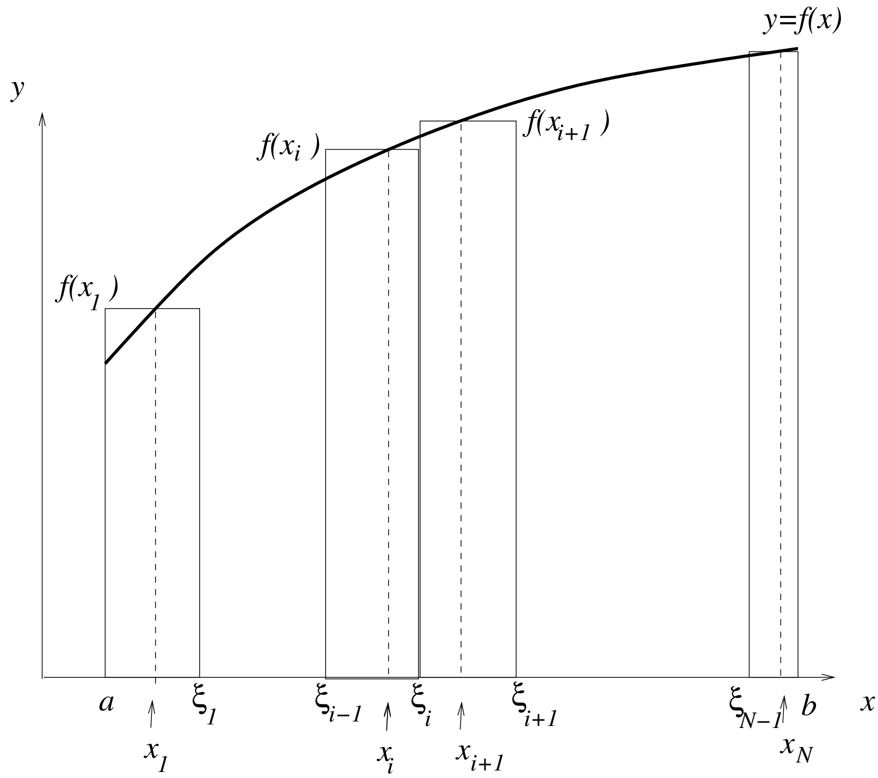

Figure 40: Approximation of the definite integral $\int_a^b f(x)\,\mathrm{d}x$.

length δx, with

$$\delta x = \frac{b-a}{N} \;, \tag{176}$$

so that the end points of the intervals are $\xi_0, \xi_1, .., \xi_N$ with

$$\begin{aligned} \xi_0 &= a \\ \xi_1 &= a + \delta x \\ \vdots & \quad \vdots \\ \xi_i &= a + i\delta x \\ \vdots & \quad \vdots \\ \xi_N &= b \;. \end{aligned} \tag{177}$$

We then choose N points $x_1, x_2, ..., x_N$, one lying in each of the subintervals, so that

$$\xi_{i-1} < x_i < \xi_i \;. \tag{178}$$

We next construct a rectangle on each subinterval, of length δx and height $f(x_i)$. The area of one rectangle is $\delta x f(x_i)$, and hence the total area of all the rectangles is

$$S = \sum_{i=1}^{N} \delta x f(x_i) \;. \tag{179}$$

The idea now is to take the limit $\delta x \to 0$, so that the length of each subinterval goes to zero while the number of subintervals goes to infinity ($N = (b-a)/\delta x \to \infty$).

The integral (if it exists) is then defined as

$$\boxed{\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\delta x \to 0} S \;,} \tag{180}$$

and it corresponds to the area between the curve $y = f(x)$ and the x axis in the range $a \leqslant x \leqslant b$.

For a given function $f(x)$, the integral (180) may not be well-defined on the interval $a \leqslant x \leqslant b$. Determining whether $f(x)$ is *integrable* can be a complicated issue, but if the function is continuous and bounded on the finite interval then we can be sure the integral converges. Note that if $y = f(x)$ is singular in $a \leqslant x \leqslant b$, then the definite integral may or may not exist.

Integrals can also have an infinite range (e.g. $b \to \infty$). Provided the integrand converges rapidly enough then the integral can be well-defined.

5.2 Integration as the inverse of differentiation

Consider the function $F(x)$ represented by the integral on an interval from a up to a variable x defined as

$$F(x) = \int_a^x f(u)\,\mathrm{d}u\;. \tag{181}$$

We can differentiate $F(x)$ from first principles using the procedure set out in section 4.1:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{\delta x\to 0} \frac{F(x+\delta x) - F(x)}{\delta x} \\ &= \lim_{\delta x\to 0} \frac{\int_a^{x+\delta x} f(u)\,\mathrm{d}u - \int_a^x f(u)\,\mathrm{d}u}{\delta x} \\ &= \lim_{\delta x\to 0} \frac{\int_x^{x+\delta x} f(u)\,\mathrm{d}u}{\delta x}\;. \end{aligned} \tag{182}$$

Now as $\delta x \to 0$, the range of integration in the numerator gets shorter and shorter and the function can be approximated as a constant over that range. In fact, we have

$$\int_x^{x+\delta x} f(u)\,\mathrm{d}u \to \delta x f(x) \quad \text{as} \quad \delta x \to 0\;. \tag{183}$$

Substituting (183) into (182) yields

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \lim_{\delta x\to 0} \frac{f(x)\delta x}{\delta x} = f(x)\;, \tag{184}$$

and we arrive at the **fundamental theorem of calculus**:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(u)\,\mathrm{d}u = f(x)\ ,} \tag{185}$$

For completeness, we state the *second fundamental theorem of calculus*: if we have a function $F(x)$, such that its derivative $F'(x) = f(x)$, then

$$\int_a^b f(u)du = F(b) - F(a)\,. \tag{186}$$

The function $f(x)$ here is assumed to be integrable on the domain $[a,\, b]$.

The integrals we have presented so far are called *definite integrals*, because they have limits (e.g. in equation 180 lower limit a and upper limit b). Note that the value of the lower limit a in (185) has no bearing on the result, so for instance

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{2a}^x f(u)\,\mathrm{d}u = f(x) \tag{187}$$

as well.

It follows that there are an infinite number of different functions $F(x)$ that we can differentiate with respect to x to give $f(x)$, but they all differ from each other only by an arbitrary additive constant.

Integrals without specific limits are called *indefinite integrals*, and for the reason given in the previous paragraph indefinite integrals are only defined up to an arbitrary additive constant.

5.3 Methods of integration

We will now review a whole series of common tricks and methods for evaluating indefinite integrals.

5.3.1 Reversal of differentiation

This is the simplest case, in which the integral can be done by inspection through prior knowledge of the appropriate derivative. For example, for the elementary functions described earlier we have the following:

$$\boxed{\begin{array}{ll} \frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1} & \Rightarrow \int x^m \,\mathrm{d}x = \frac{x^{m+1}}{m+1} + c \text{ (if } \quad m \neq -1) \\ \\ \frac{\mathrm{dln}x}{\mathrm{d}x} = \frac{1}{x} & \Rightarrow \int \frac{1}{x} \,\mathrm{d}x = \mathrm{ln}x + c \\ \\ \frac{\mathrm{d}\exp(mx)}{\mathrm{d}x} = m\exp(mx) & \Rightarrow \int \exp(mx) \,\mathrm{d}x = \frac{\exp(mx)}{m} + c \;, \end{array}} \tag{188}$$

while for trigonometric and hyperbolic functions we have

$$\boxed{\begin{array}{ll} \frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x & \Rightarrow \int \cos x \,\mathrm{d}x = \sin x + c \\ \\ \frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x & \Rightarrow \int \sin x \,\mathrm{d}x = -\cos x + c \\ \\ \frac{\mathrm{d}\tan x}{\mathrm{d}x} = \sec^2 x & \Rightarrow \int \sec^2 x \,\mathrm{d}x = \tan x + c \\ \\ \frac{\mathrm{d}\sinh x}{\mathrm{d}x} = \cosh x & \Rightarrow \int \cosh x \,\mathrm{d}x = \sinh x + c \\ \\ \frac{\mathrm{d}\cosh x}{\mathrm{d}x} = \sinh x & \Rightarrow \int \sinh x \,\mathrm{d}x = \cosh x + c \\ \\ \frac{\mathrm{d}\tanh x}{\mathrm{d}x} = \mathrm{sech}^2 x & \Rightarrow \int \mathrm{sech}^2 x \,\mathrm{d}x = \tanh x + c \;. \end{array}} \tag{189}$$

5.3.2 Inverse trigonometric and hyperbolic functions

To differentiate $y = \sinh^{-1}(x/a)$ we first write $\sinh y = x/a$. Next we differentiate with respect to x

$$\begin{aligned}
\frac{\mathrm{d}(\sinh y)}{\mathrm{d}x} &= \frac{1}{a} \\
\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}\cosh y &= \frac{1}{a} \\
\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a\cosh y} \\
&= \frac{1}{a\sqrt{\sinh^2 y + 1}} \\
&= \frac{1}{\sqrt{a^2+x^2}} \, .
\end{aligned}$$

In a similar way we can show that

$$\frac{\mathrm{d}(\cosh^{-1}(x/a))}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-a^2}} \, . \tag{190}$$

These two results lead us to the standard integrals

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a) + c \, , \tag{191}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + c \, . \tag{192}$$

The inverse hyperbolic integrals (191) and (192) should be contrasted with the corresponding results for the trigonometric functions, e.g.

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c \, . \tag{193}$$

Note that $\cos^{-1}(x/a) = \pi/2 - \sin^{-1}(x/a)$.

5.3.3 Integrands of form $[f(x)]^\alpha \, \mathrm{d}f/\mathrm{d}x$

The following integral can be completed directly

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}[f(x)]^\alpha \, \mathrm{d}x = \frac{1}{\alpha+1}[f(x)]^{\alpha+1} \qquad (194)$$

when $\alpha \neq -1$. This is the inverse of the chain rule (159). We will give some examples:

Example 5.1 Complete the following: (a) $\int \cos x \sin^3 x \, dx$, (b) $\int (\tanh^6 x)\mathrm{sech}^2 x \, dx$, (c) $\int x \exp(-x^2) dx$.

(a)

$$\int \cos x \sin^3 x dx = \int \frac{1}{4}\frac{d \sin^4 x}{dx} \, dx = \frac{1}{4}\sin^4 x + c.$$

(b) Remember that $d \tanh x/dx = \mathrm{sech}^2 x$. Now we can do the following:

$$\int \tanh^6 x \, \mathrm{sech}^2 x dx = \int \frac{1}{7}\frac{d \tanh^7 x}{dx} \, dx = \frac{1}{7}\tanh^7 x + c.$$

(c)

$$\int x e^{-x^2} dx = \int -\frac{1}{2}\frac{d e^{-x^2}}{dx} \, dx = -\frac{1}{2}e^{-x^2} + c.$$

The result (194) does not work when $\alpha = -1$, but it can be replaced by

$$\int \frac{\mathrm{d}f}{\mathrm{d}x}\frac{1}{f(x)} \, \mathrm{d}x = \ln[f(x)] + c \, . \qquad (195)$$

Example 5.2 Complete the following: (a) $\int x/(x^2+1)$, (b) $\int \tanh x$.

$$(a) \int \frac{x}{x^2+1} dx = \int \frac{1}{2}\frac{d \ln(x^2+1)}{dx} \, dx = \frac{1}{2}\ln(x^2+1) + c.$$

(b)
$$\int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{\cosh x} \frac{d \cosh x}{dx} dx,$$
$$= \int \frac{d}{dx} \left[\ln(\cosh x)\right] dx,$$
$$= \ln(\cosh x) + c.$$

5.3.4 Powers of trigonometric functions

Trigonometric identities are often useful. For instance, we can use the identities

$$\begin{array}{rclcrcl} \cos 2x &=& 1 - 2\sin^2 x &\Rightarrow& \sin^2 x &=& \frac{1}{2}[1 - \cos 2x] \\ \cos 2x &=& 2\cos^2 x - 1 &\Rightarrow& \cos^2 x &=& \frac{1}{2}[1 + \cos 2x] \end{array} . \tag{196}$$

Then we can evaluate

$$\begin{aligned} \int \sin^4 x \, \mathrm{d}x &= \int \frac{1}{4}(1 - \cos 2x)^2 \, \mathrm{d}x \\ &= \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \, \mathrm{d}x \\ &= \int \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}[\cos 4x + 1]\right) , \end{aligned} \tag{197}$$

where the last step has been accomplished using (196) but with x replaced by $2x$. Each term in the integrand of (197) is now of an elementary form, and can be evaluated to give

$$\begin{aligned} \int \sin^4 x \, \mathrm{d}x &= \frac{1}{4}\left(x - \frac{2}{2}\sin 2x + \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]\right) + c \\ &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c\,. \end{aligned} \tag{198}$$

Odd powers can often be handled using the method outlined in Section 5.3.3. For example:

$$\int \sin^3 x \, \mathrm{d}x = \int \sin x \sin^2 x \, \mathrm{d}x = \int \sin x(1 - \cos^2 x) \, \mathrm{d}x = -\cos x + \frac{\cos^3 x}{3} + c\,.$$

(199)

Example 5.3 Complete the following: (a) $\int \cos^4 x\, dx$, (b) $\int \tan^5 x\, dx$.

$$\begin{aligned}
\int \cos^4 x dx &= \frac{1}{4}\int (1+\cos 2x)^2 dx, \\
&= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x\, dx, \\
&= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\ dx, \\
&= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.
\end{aligned}$$

$$\begin{aligned}
\int \tan^5 x dx = \int \tan^3 x\ \tan^2 x\, dx &= \int \tan^3 x(\sec^2 x - 1)dx, \\
&= \int \tan^3 x \sec^2 x - \tan x\ \tan^2 x\ dx, \\
&= \frac{1}{4}\tan^4 x - \int \tan x(\sec^2 x - 1)dx, \\
&= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x) + c
\end{aligned}$$

5.3.5 Partial fractions

Consider the integral

$$\int \frac{1}{x^2+x}\, \mathrm{d}x\ . \tag{200}$$

We can make progress if we split the integrand into its partial fractions, i.e.

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{\alpha}{x} + \frac{\beta}{x+1} \tag{201}$$

for constants α and β. To find α and β we write

$$\frac{\alpha}{x} + \frac{\beta}{x+1} = \frac{\alpha(x+1)+\beta x}{x(x+1)} = \frac{x(\alpha+\beta)+\alpha}{x(x+1)}\,, \tag{202}$$

and then comparing the numerator of the final expression in (202) with the integrand in (200) we see that $\alpha+\beta=0$ and $\alpha=1$ (implying $\beta=-1$), so that

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}\,. \tag{203}$$

The integral (200) then becomes

$$\begin{aligned}\int \frac{1}{x^2+x}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x - \int \frac{1}{x+1}\,\mathrm{d}x \\ &= \ln x - \ln(x+1) + c\,. \end{aligned} \tag{204}$$

<u>Example 5.4</u> Evaluate

$$\int \frac{x+1}{1-x+x^2-x^3}\,\mathrm{d}x\,. \tag{205}$$

We re-express the integrand in partial fractions, first factorising the denominator:

$$\begin{aligned}\frac{x+1}{1-x+x^2-x^3} = \frac{x+1}{(1-x)(1+x^2)} &= \frac{A}{1-x} + \frac{Bx+C}{1+x^2}, \\ &= \frac{A+Ax^2+Bx+C-Bx^2-Cx}{(1-x)(1+x^2)}, \\ &= \frac{A+C+(B-C)x+(A-B)x^2}{(1-x)(1+x^2)},\end{aligned}$$

where A, B, and C are constants to be determined. In the numerators of the left and right sides, the coefficients of 1, x, and x^2 can be equated giving the equations:

$$A+C=1, \quad B-C=1, \quad A-B=0.$$

These can be solved quickly to give us $A = B = 1$ and $C = 0$. We are now in a good place to evaluate the integral:

$$\int \frac{x+1}{1-x+x^2+x^3}\,dx = \int -\frac{1}{x-1} + \frac{x}{1+x^2}\,dx = -\ln(x-1) + \tfrac{1}{2}\ln(1+x^2) + c.$$

5.3.6 Trigonometric and other substitutions

Difficult integrals can often be simplified by changing variables. The trick is to know which substitution to use! The main point is to bring the integral to a recognised form where the integral can be done by inspection using known results. This technique is best developed by doing lots of different examples.

Consider the exponential integral

$$\int xe^{-x^2} dx. \tag{206}$$

Setting $u = x^2$ and thus $du = 2x\,dx$ gives

$$\int xe^{-x^2} dx = \frac{1}{2}\int e^{-u} du = -\frac{1}{2}e^{-u} + \text{const.} = -\frac{1}{2}e^{-x^2} + \text{const.} \tag{207}$$

It is important in a definite integral to also change the limits to match the new variable u.

Commonly, one can employ substitutions of trigonometric functions. Consider

$$\int \frac{1}{1+x^2}\,\mathrm{d}x\ . \tag{208}$$

Try $x = \tan t$, so differentiating

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t\ ,$$

which allows the replacement in (208)

$$\mathrm{d}x = \sec^2 t\ \mathrm{d}t\ . \tag{209}$$

In this way

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{1}{1+\tan^2 t}\,\sec^2\mathrm{t}\;\mathrm{d}t = \int 1\;\mathrm{d}t\;, \tag{210}$$

where the last step has been accomplished simply via the identity $\tan^2 t + 1 = \sec^2 t$.

The integral on the right of (210) is now very easy indeed, and we find that

$$\boxed{\int \frac{1}{1+x^2}\,\mathrm{d}x = t + c = \tan^{-1} x + c\;.} \tag{211}$$

When encountering integrands with quadratics in the denominator, we can often *complete the square* in order to make the trigonometric substitutions above.

As an example, consider the following integral

$$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \int \frac{du}{\sqrt{4-u^2}}\,, \tag{212}$$

where we have substituted $u = x - 1$ and $du = dx$. Now substitute $u = 2\sin\,t$ with $du = 2\cos\,t\,dt$ to find

$$\int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos t\,dt}{2\cos t} = t + \text{const.} = \sin^{-1}\left(\frac{x-1}{2}\right) + \text{const.} \tag{213}$$

Another useful trick when dealing with integrals of awkward trigonometric functions is to use the *half-angle formula*.

Start with the substitution

$$\tan(x/2) = t\;. \tag{214}$$

Then we can show that

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2}{\mathrm{cosec}(x/2)\sec(x/2)} = \frac{2}{\sqrt{1+t^{-2}}\sqrt{1+t^2}} = \frac{2t}{1+t^2}\;, \tag{215}$$

Similarly,

$$\cos x = 2\cos^2(x/2) - 1 = \frac{2}{\sec^2(x/2)} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}\,, \qquad (216)$$

and

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1-t^2}\,. \qquad (217)$$

Finally we need a way to re-express dx in terms of dt. We have

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) = \frac{1}{2}(1+t^2).$$

Thus

$$dx = \frac{2}{1+t^2}\,dt. \qquad (218)$$

Example 5.5 Use the substitution $t = \tan(x/2)$ to evaluate $\int \sec x \,\mathrm{d}x$.

$$\int \sec x dx = \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}\,dt = \int \frac{2}{1-t^2}\,dt.$$

We now use partial fractions to re-express the integrand. Skipping a few steps, which you should supply,

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}.$$

Now we can evaluate the integral:

$$\int \sec x\,dx = \int \frac{1}{1-t} + \frac{1}{1+t}\,dt = -\ln(1-t) + \ln(1+t) + c.$$
$$= \ln\left(\frac{1+t}{1-t}\right) + c = \ln\left(\frac{1+\tan(x/2)}{1-\tan(x/2)}\right) + c.$$

5.3.7 Integration by parts

Integration by parts is closely related to the product rule, which we now rearrange and then integrate between the limits $a \leqslant x \leqslant b$:

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}(uv)}{\mathrm{d}x}\,\mathrm{d}x - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x\,. \tag{219}$$

The first integral on the right hand side can be completed (using the fundamental theorem of calculus), and we are then left with the rule for **integration by parts**

$$\boxed{\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \Big[uv\Big]_a^b - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x\,.} \tag{220}$$

Example 5.6 Evaluate $\int \ln x\,\mathrm{d}x$ and $\int x\sec^2 x\,\mathrm{d}x$.

$$\int \ln x dx = \int \frac{dx}{dx}\ln x dx = x\ln x - \int x\frac{d\ln x}{dx}\,dx,$$
$$= x\ln x - \int 1\,dx = x\ln x - x + c.$$

$$\int x\sec^2 x\,dx = \int x\frac{d\tan x}{dx}\,dx = x\tan x - \int \frac{dx}{dx}\tan x\,dx,$$
$$= x\tan x - \int \tan x dx = x\tan x + \ln(\cos x) + c.$$

5.3.8 Symmetry - integrating even and odd functions

When a function is described as being *even* or *odd* then this refers to its symmetry or antisymmetry about the y-axis. Specifically

$$\begin{aligned} f(x) &= f(-x) \quad \textbf{even function,} \\ f(x) &= -f(-x) \quad \textbf{odd function,} \end{aligned} \tag{221}$$

For example $f(x) = \cos x$ is an even function, $f(x) = \sin x$ is an odd function. Note that for an odd function $f(0) = -f(-0)$, so that $f(0) = 0$. Note also that most functions are neither odd nor even!

Evenness or oddness can be important when integrating.

For instance, without doing detailed calculations we can see straight away that

$$\int_{-\pi/4}^{\pi/4} \frac{x}{1+x^2}\,\mathrm{d}x = 0\,. \tag{222}$$

The reason for this is that the integrand in (222) is an **odd** function of x, so that the area under the curve for $x > 0$ exactly cancels out with the area under the curve for $x < 0$, to give a total area of zero.

Another example, this time involving infinite limits, is

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,\mathrm{d}x = 0\,, \tag{223}$$

because again the integrand is an **odd** function of x.

In both cases this works because the integration interval itself is symmetric around $x = 0$. If we integrated between $x = 1$ and $x = 2$ (for example) then the integrals above would not necessarily be zero.

Of course, for **even** functions things are different because now the areas on either side of $x = 0$ add up rather than cancel. For example,

$$\int_{-1}^{1} \frac{x^2}{1+x^2}\,\mathrm{d}x = 2\int_{0}^{1} \frac{x^2}{1+x^2}\,\mathrm{d}x\,, \tag{224}$$

while

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x = 2\int_{0}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x\,. \tag{225}$$

In (224) the integral on the right hand side can be calculated using the substitution $x = \tan y$, to give

$$\begin{aligned} 2\int_0^1 \frac{x^2}{1+x^2}\,\mathrm{d}x &= 2\int_0^{\pi/4} \frac{\tan^2 y}{1+\tan^2 y}\sec^2 y\,\mathrm{d}y = 2\int_0^{\pi/4} \tan^2 y\,\mathrm{d}y \\ &= 2\int_0^{\pi/4} (\sec^2 y - 1)\,\mathrm{d}y \\ &= 2\Big[\tan y - y\Big]_0^{\pi/4} = 2 - \frac{\pi}{2}\,. \end{aligned} \tag{226}$$

Unfortunately, the integral on the right hand side in (225) requires more advanced methods beyond the scope of this course (cf. the Cauchy residue theorem).

5.3.9 Reduction formulae

Reduction formulae are often used to reduce a complicated integral down to something more manageable.

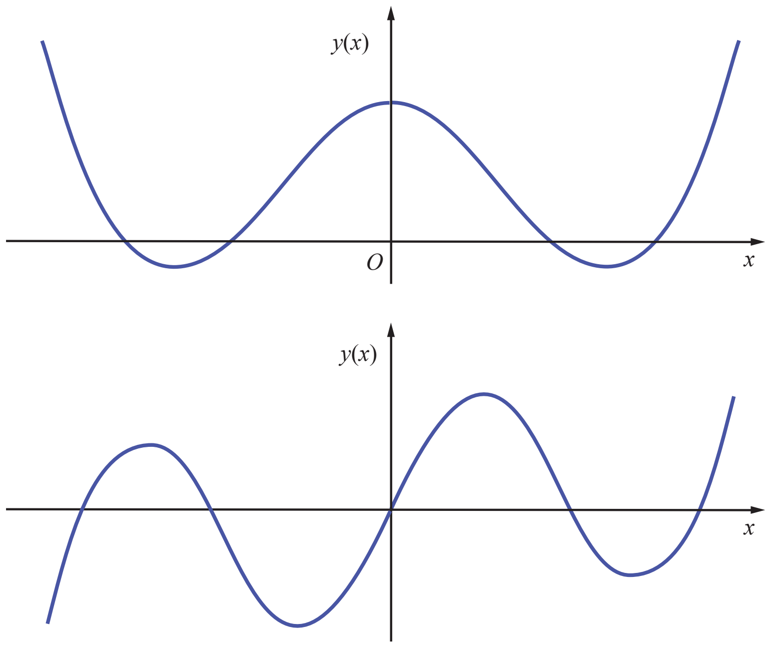

Figure 41: Even (top) and odd functions.

For instance, suppose we wanted to know $\int_0^{\pi/2} \sin^{10} x\, \mathrm{d}x$ or $\int_0^{\pi/2} \sin^{1000} x\, \mathrm{d}x$. Before tackling these individually, consider:

$$I_{2n} \equiv \int_0^{\pi/2} \sin^{2n} x \; \mathrm{d}x \tag{227}$$

for n a positive integer.

The idea is to relate I_{2n} to a similar integral with a lower power of $\sin x$. We write

$$I_{2n} = \int_0^{\pi/2} \sin x \sin^{2n-1} x \; \mathrm{d}x \; , \tag{228}$$

and then use integration by parts (with $u = \sin^{2n-1} x$ and $\mathrm{d}v/\mathrm{d}x = \sin x$) to get

$$\begin{aligned}
\int_0^{\pi/2} \sin x \sin^{2n-1} x \; \mathrm{d}x &= \Big[-\cos x \sin^{2n-1} x \Big]_0^{\pi/2} + (2n-1) \int_0^{\pi/2} \cos^2 x \sin^{2n-2} x \; \mathrm{d}x \\
&= (2n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{2n-2} x \; \mathrm{d}x \\
&= (2n-1) \int_0^{\pi/2} \sin^{2n-2} x \; \mathrm{d}x - (2n-1) \int_0^{\pi/2} \sin^{2n} x \; \mathrm{d}x \; .
\end{aligned}$$

In other words, we have

$$I_{2n} = (2n-1) I_{2n-2} - (2n-1) I_{2n} \; , \tag{229}$$

or rearranging

$$I_{2n} = \frac{(2n-1)}{2n} I_{2n-2} \; . \tag{230}$$

Equation (230) is a *recurrence relation*, which means we can just apply it over and over again, so

$$\begin{aligned} I_{2n} &= \frac{(2n-1)}{2n} I_{2n-2} \\ &= \frac{(2n-1)}{2n} . \frac{(2n-3)}{2n-2} I_{2n-4} \\ &= \frac{(2n-1)}{2n} . \frac{(2n-3)}{2n-2} . \frac{(2n-5)}{2n-4} I_{2n-6} \\ &= \dots . \end{aligned} \tag{231}$$

Note how the index on the integral goes down by 2 each time. If we do this operation n times then the index of the integral goes down to zero and we end up with

$$I_{2n} = \frac{(2n-1)(2n-3)(2n-5)... \times 3 \times 1}{(2n)(2n-2)(2n-4)... \times 4 \times 2} I_0 \ . \tag{232}$$

The point now is that I_0 is very easy to calculate, because

$$I_0 = \int_0^{\pi/2} \sin^0 x \ \mathrm{d}x = \int_0^{\pi/2} 1 \ \mathrm{d}x = \pi/2 \ , \tag{233}$$

and putting (233) back into (232) gives us a closed expression for I_{2n}. Therefore, back to our original query, we have $\int_0^{\pi/2} \sin^{10} x \, \mathrm{d}x = (63/512)\pi$. (We also get $\int_0^{\pi/2} \sin^{1000} x \, \mathrm{d}x = 0.0396234 \ldots$.)

<u>Example 5.7</u> [2005, paper 2, question 11F]

(a) By expressing the integrand in partial fractions, evaluate

$$\int_1^2 \frac{3x^2 + 5x + 1}{x(x+1)(x+2)} \ \mathrm{d}x \ .$$

(b) Evaluate the definite integral

$$\int_0^{\pi/4} \frac{1}{1 + \cos 2\theta} \ \mathrm{d}\theta \ .$$

(c) Using your results from (b), or otherwise, evaluate the definite integral

$$\int_0^{\pi/2} \frac{1}{1+\sin\phi}\, \mathrm{d}\phi \,.$$

First let us rewrite the integrand in terms of several fractions:

$$\begin{aligned} \frac{3x^2+5x+1}{x(x+1)(x+2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}, \\ &= \frac{A(x+1)(x+2) + Bx(x+2) + Cx(x+1)}{x(x+1)(x+2)}, \\ &= \frac{(A+B+C)x^2 + (3A+2B+C)x + 2A}{x(x+1)(x+2)}. \end{aligned}$$

Equating the coefficients of x^2, x, and 1 on the numerators of both sides of the equation gives us three equations for A, B, and C:

$$A+B+C=3, \quad 3A+2B+C=5, \quad 2A=1.$$

We see that $A = 1/2$ straightaway. Subtracting the first two equations from each other gives $B = 1$, and then $C = 3/2$ follows. Now we go to the integral itself:

$$\begin{aligned} \int_1^2 \frac{3x^2+5x+1}{x(x+1)(x+2)} dx &= \int_1^2 \frac{1/2}{x} + \frac{1}{x+1} + \frac{3/2}{x+2}\, dx, \\ &= \left[\frac{1}{2}\ln x + \ln(x+1) + \frac{3}{2}\ln(x+2)\right]_1^2, \\ &= \frac{1}{2}(\ln 2 - \ln 1) + \ln 3 - \ln 2 + \frac{3}{2}(\ln 4 - \ln 3), \\ &= \ln\sqrt{\frac{32}{3}}. \end{aligned}$$

(b) Recall that $\cos 2\theta = 2\cos^2\theta - 1$, so our integral becomes

$$\begin{aligned} \int_0^{\pi/4} \frac{d\theta}{1+\cos 2\theta} = \int_0^{\pi/4} \frac{d\theta}{2\cos^2\theta} &= \frac{1}{2}\int_0^{\pi/4} \sec^2\theta\, d\theta, \\ &= \frac{1}{2}\left[\tan\theta\right]_0^{\pi/4} = \frac{1}{2}\tan(\pi/4) = \frac{1}{2}. \end{aligned}$$

(c) Obviously, we would like to transform in some way $\sin\phi$ into $\cos 2\theta$. Recall that $\sin(\pi/2 - x) = \cos x$, so how about the substitution:

$$\phi = \frac{\pi}{2} - 2\theta\,?$$

Then $d\phi/d\theta = -2$, which means $d\phi = -2d\theta$. The integration limits need changing too: from $[0, \pi/2]$ to $[\pi/4, 0]$. We then have

$$\int_0^{\pi/2} \frac{d\phi}{1+\sin\phi} = \int_{\pi/4}^{0} \frac{-2d\theta}{1+\sin(\pi/2-2\theta)} = 2\int_0^{\pi/4} \frac{d\theta}{1+\cos 2\theta} = 1,$$

where the last equality comes about because of part (b).