2 Complex numbers

Though they don't exist, *per se*, complex numbers pop up all over the place when we do science. They are an especially useful mathematical tool in problems involving:

- oscillations and waves in light, fluids, magnetic fields, electrical circuits, etc.,
- stability problems in fluid flow and structural engineering,
- signal processing (the Fourier transform, etc.),
- quantum physics, e.g. Schroedinger's equation,

  $$i\frac{\partial \psi}{\partial t} + \nabla^2 \psi + V(x)\psi = 0\,,$$

  which describes the wavefunctions of atomic and molecular systems,
- difficult differential equations.

For your revision: recall that the general quadratic equation, $ax^2 + bx + c = 0$ (solving for x), has two solutions given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,. \tag{85}$$

But what happens when the discriminant is negative?
For instance, consider

$$x^2 - 2x + 2 = 0\,. \tag{86}$$

Its solutions are

$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{2 \pm \sqrt{-4}}{2} \\ &= 1 \pm \sqrt{-1}\,. \end{aligned}$$

These two solutions are evidently *not* 'normal' numbers!

If we put that aside, however, and accept the existence of the square root of minus one, written as

$$i = \sqrt{-1}\,, \tag{87}$$

then these two solutions, $1+i$ and $1-i$, can be assigned to the set of *complex numbers*. The quadratic equation now can be factorised as

$$z^2 - 2z + 2 = (z - 1 - i)(z - 1 + i) = 0\,,$$

and to be able to factorise every polynomial is very useful, as we shall see.

2.1 Complex algebra

2.1.1 Definitions

A complex number z takes the form

$$\boxed{z = x + iy\ ,} \tag{88}$$

where x and y are real numbers and i is the imaginary unit satisfying

$$i^2 = -1\,. \tag{89}$$

We call x and y the real and imaginary parts of z respectively, and write

$$\boxed{\begin{array}{lllll} x & = & \Re(z) & \text{or} \quad \mathrm{Re}(z) & \text{the real part of } z\,, \\ y & = & \Im(z) & \text{or} \quad \mathrm{Im}(z) & \text{the imaginary part of } z\,. \end{array}}$$

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, i.e. $z_1 = z_2$, then their real and imaginary parts must be equal, that is, both $x_1 = x_2$ and $y_1 = y_2$.

2.1.2 Addition

The sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is also a complex number given by

$$\boxed{z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)\,.} \tag{90}$$

The real part is $\Re(z_1 + z_2) = x_1 + x_2$ and the imaginary part is $\Im(z_1 + z_2) = y_1 + y_2$. The commutativity and associativity of real numbers under addition is therefore also passed on to the complex numbers, e.g. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

2.1.3 Multiplication

We can multiply complex numbers, provided we know how to multiply i by itself. The following table gives the results for the powers of i:

$$\boxed{\begin{array}{ccccccc} i & = & i & & & & \\ i^2 & = & -1 & & & & \\ i^3 & = & i^2 \times i & = & -i & & \\ i^4 & = & i^3 \times i & = & -i \times i & = & 1 \\ i^5 & = & i^4 \times i & = & i & & \\ \vdots & & \vdots & & \vdots & & \end{array}} \tag{91}$$

Note the pattern has a fourfold periodicity with $i^{4n+m} = i^m$, where n is any integer.

Okay, now consider the product of the two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= x_1x_2 + i(x_1y_2 + y_1x_2) - y_1y_2 \end{aligned}$$

where we have used (89). So, collecting real and imaginary parts, we have

$$\boxed{z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)\,.} \tag{92}$$

Complex multiplication inherits commutativity and associativity from the real numbers,

$$z_1 z_2 = z_2 z_1\,, \qquad z_1(z_2 z_3) = (z_1 z_2) z_3\,, \tag{93}$$

and it is also distributive

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3\,. \tag{94}$$

Example 2.1 If $z_1 = 3 + i$ and $z_2 = 1 - i$ calculate $z_1 + z_2$, $z_1 - z_2$ and $z_1 z_2$.

$$\begin{aligned} z_1 + z_2 &= 3 + i + 1 - i = 4, \\ z_1 - z_2 &= 3 + i - (1 - i) = 3 - 1 + (1 + 1)i = 2 + 2i, \end{aligned}$$

$$\begin{aligned} z_1\, z_2 = (3 + i)(1 - i) = 3 - 3i + i + i(-i) &= 3 - 2i - i^2, \\ &= 3 - 2i + 1, \\ &= 4 - 2i. \end{aligned}$$

2.1.4 Complex conjugate and modulus

The complex conjugate of $z = x + iy$ is found by changing the sign of its imaginary component. It is denoted by z^* (or often $\overline{z}$) and defined by

$$\boxed{z^* = x - iy\,.} \tag{95}$$

Note that it follows that we must have $z + z^* = 2\Re(z)$ and $z - z^* = 2i\Im(z)$.

If we take the product of z with its complex conjugate z^* we find

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 + i(-xy + xy) = x^2 + y^2 \,,$$

which is real and non-negative. Thus

$$\boxed{zz^* = x^2 + y^2 \,.} \tag{96}$$

The *modulus* of z, denoted by $|z|$ or mod(z), is defined by

$$\boxed{|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2} \,.} \tag{97}$$

2.1.5 Division

It is easiest to compute the division of one complex number by another by using the properties of the conjugate and modulus.

The division of z_1 by z_2 may be manipulated into

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \,, \tag{98}$$

and the denominator is conveniently a real number. Then we use the rule for multiplication of complex numbers, in the numerator, to compute the result.

The general rule for simplifying expressions involving division by a complex number z_2 is to multiply numerator and denominator by the complex conjugate z_2^*/z_2^* (i.e. the identity), thus making the denominator real $z_2 z_2^* = |z_2|^2$.

Example 2.2 Take $z_1 = 3 + i$ and $z_2 = 1 - i$ again and calculate z_1/z_2.

$$\frac{z_1}{z_2} = \frac{3+i}{1-i} \times \frac{1+i}{1+i} = \frac{3 + 3i + i + i^2}{1^2 + 1^2} = 1 + 2i.$$

2.2 The Complex Plane

2.2.1 Argand diagram

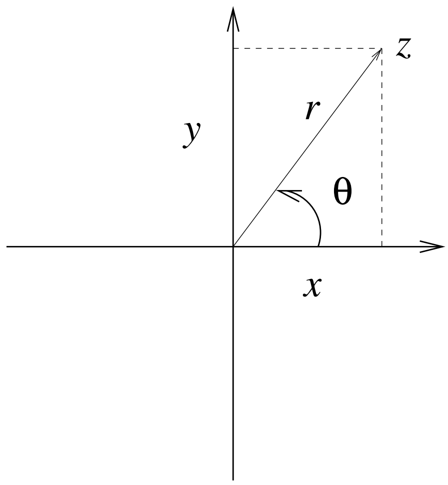

Figure 27: The Argand diagram of the complex plane.

The real (x) and imaginary (y) parts of a complex number z are independent quantities, so we can think of z as plotting a point in a two-dimensional space, (x, y), where the y axis corresponds to the imaginary part and the x axis corresponds to the real part of the number.

In fact, we can go further and think of a complex number as a two-dimensional vector.

This two-dimensional space is often called the Argand diagram.

Now purely algebraic processes (conjugation, addition, division, etc.) can be represented as geometric operations. For instance

- The complex conjugate z^* is found by merely reflecting any point z about the real (i.e. x) axis.
- Addition and subtraction of complex numbers is essentially equivalent to vector addition and subtraction.

2.2.2 Polar form of complex numbers

We can also use plane polar coordinates in the Argand plane. Simple trigonometry gives us

$$\boxed{r = \sqrt{x^2 + y^2} = |z|\,,} \tag{99}$$

The radius r is, in fact, the modulus of z, met earlier.

The polar angle θ is called the *argument* or *phase* of z. It is the angle subtended by the real axis and the line made by the complex number and the origin. The following formula can help us get the argument

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)\,.} \tag{100}$$

(as can $\cos\theta = x/r$ or $\sin\theta = y/r$.) However, care must be taken because we need to first determine in which quadrant the complex number lies (the above formula does not distinguish between the first and third quadrant, or between the second and fourth).

It is conventional to restrict the argument θ to the range $-\pi < \theta \leqslant \pi$ in order to make it unique; this is called the *principal range*, and then θ is the *principal argument*.

The polar representation of z is:

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta)\,.} \tag{101}$$

For the complex conjugate, it is clear that the argument will become $-\theta$, i.e.,

$$z^* = x - iy = r(\cos\theta - i\sin\theta) = r\left[\cos(-\theta) + i\sin(-\theta)\right]\,. \tag{102}$$

The inverse, on the other hand, is

$$z^{-1} = \frac{z^*}{|z|^2} = r^{-1}\left[\cos(-\theta) + i\sin(-\theta)\right] \tag{103}$$

Example 2.3 Calculate the modulus and argument for $z_1 = 3+i$ and $z_2 = 1-i$.

For z_1, we have $r = |z_1| = \sqrt{3^2+1^2} = \sqrt{10}$, while $\theta = \tan^{-1}(1/3) \approx 18.43°$. For z_2, $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$, and $\theta = \tan^{-1}(-1) = -\pi/4$.

Example 2.4 What shape in the Argand diagram is described by the equation $3|z| = |z-i|$? What about $|z| = |z-i|$?

First set $z = x+iy$. Squaring the first equality yields

$$
\begin{aligned}
&9|z|^2 = |z-i|^2, \\
&9(x^2+y^2) = |x+i(y-1)|^2 = x^2+(y-1)^2 = x^2+y^2-2y+1, \\
&\to 8x^2+8y^2+2y = 1, \\
&x^2+y^2+\tfrac{1}{4}y = \tfrac{1}{8}, \\
&x^2+(y+\tfrac{1}{8})^2 = (\tfrac{3}{8})^2
\end{aligned}
$$

So in the Argand diagram we have a circle centred at $(0,-\frac{1}{8})$ (or $z=-i/8$) with radius $\frac{3}{8}$.

Let us also square the second equality:

$$
\begin{aligned}
&|z|^2 = |z-i|^2 \\
&x^2+y^2 = x^2+(y-1)^2 = x^2+y^2-2y+1, \\
&\to y = \tfrac{1}{2}.
\end{aligned}
$$

Since x is arbitrary, $z = x+\frac{1}{2}i$, a straight line. Points on the line are equidistant from $z=0$ and $z=i$.

2.3 The complex exponential

2.3.1 Euler's formula

There is a profound relationship between trigonometric functions and the complex exponential function:

$$\cos\theta + i\sin\theta = e^{i\theta}\,. \tag{104}$$

We will prove this later. This is called Euler's formula.

Euler's formula means that we can write any complex number compactly as

$$\boxed{z = r\,e^{i\theta}\,.} \tag{105}$$

There is some degeneracy here because we can add any integer multiple of 2π onto θ without changing the value of z. (This follows from $\cos(\theta+2\pi n) = \cos\theta$ and $\sin(\theta + 2\pi n) = \sin\theta$, for integer n.) A consequence is the useful identity:

$$\exp(2i\pi n) = 1,$$

for integer n.

2.3.2 Multiplication

The exponential form makes it *easy* to do multiplication and division. If $z_1 = r_1\exp(i\theta_1)$ and $z_2 = r_2\exp(i\theta_2)$ then

$$\boxed{z_1 z_2 = r_1 r_2 \exp\left(i(\theta_1 + \theta_2)\right)\,.} \tag{106}$$

So when multiplying complex numbers the moduli are *multiplied* together and the arguments are *added*.

2.3.3 Division

We looked at division previously (98) and this now becomes

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\exp(i[\theta_1 - \theta_2])\,. \tag{107}$$

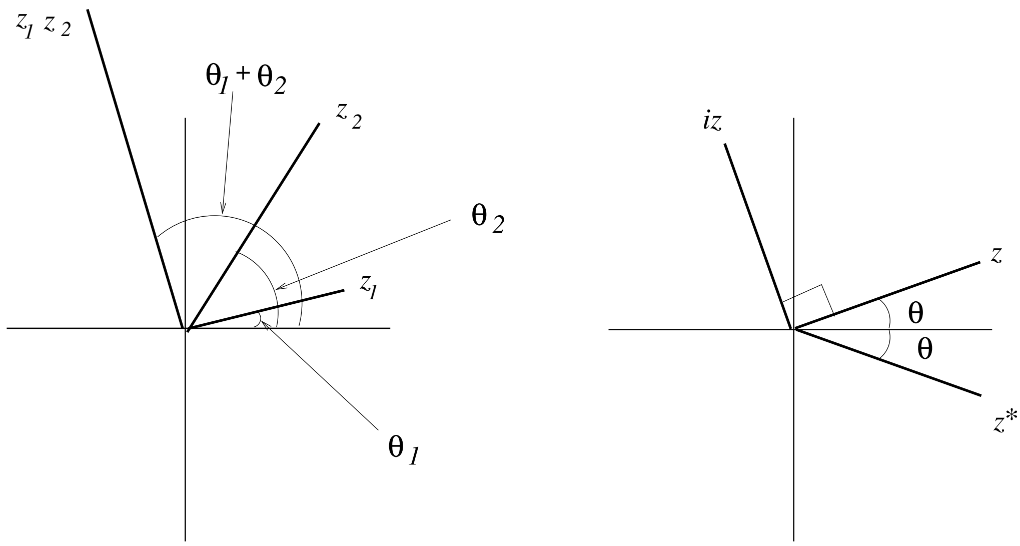

Figure 28: Geometrical interpretation of (a) multiplication of z_1 and z_2; (b) multiplication by i and complex conjugation.

So when dividing complex numbers the moduli are divided and the arguments are subtracted.

2.3.4 Geometric manifestations

A geometrical interpretation of multiplication of z_1 by z_2 corresponds to rotation of z_1 by the argument of z_2 and a scaling of z_1's modulus by $|z_2|$. Note the special case of multiplication by i corresponds simply to rotation by 90° anti-clockwise. Taking the complex conjugate of z corresponds to reflection in the x axis.

2.3.5 New expressions for $\cos$ and $\sin$

Taking the complex conjugate of (104) yields

$$\exp(-i\theta) = \cos\theta - i\sin\theta \ , \tag{108}$$

and adding (104) to (108) gets us

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\ .} \tag{109}$$

Similarly, subtracting (108) from (104) gives an expression for sin

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\ .} \tag{110}$$

2.3.6 Fundamental Theorem of Algebra

We state the fundamental theorem of algebra without proof. The polynomial equation of degree n (a positive integer)

$$a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0\,, \qquad a_n \neq 0\,. \tag{111}$$

has n complex roots for any possible (complex) coefficients $a_0, a_1, ...a_n$.

2.3.7 Roots of unity

We want to solve the equation

$$z^n = 1\ , \tag{112}$$

where n is an integer. One root is $z = 1$, of course, but (112) is a polynomial equation of degree n, so we expect n roots!

The way to grab these is to use the complex exponential notation and allow for degeneracy in the complex argument.

- We recognise that 1 is a complex number with modulus 1 and argument $0 + 2\pi m$, for m an integer:

$$z^n = 1 = \exp(2\pi i m) \qquad m = 0, 1, 2, \ldots\ldots\,. \tag{113}$$

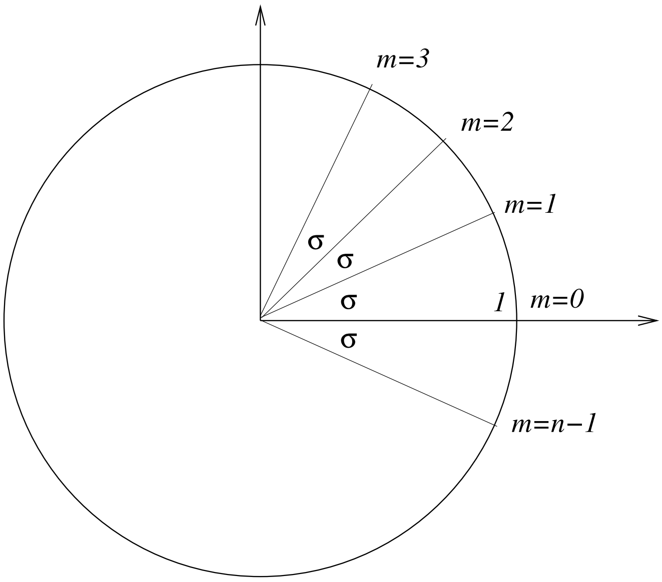

Figure 29: The roots of $z^n = 1$ in the complex plane, given by (115). The angle $\sigma = 2\pi/n$.

- then take the nth root of both sides

$$z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, ... \quad . \tag{114}$$

However, the root with $m = n$ is $z = \exp(2\pi i) = 1$, which is the same as the root with $m = 0$, so the n *distinct* roots of equation (112) are

$$\boxed{z = \exp(2\pi i m/n) \qquad m = 0, 1, 2, ...n - 1\ .} \tag{115}$$

Alternatively, we can write $\omega = e^{2\pi i/n}$, and the n roots are 1, ω, ω^2, ..., ω^{n-1}. These roots are distributed around the unit circle (with radius 1 centred on the origin) at regular angles of $2\pi/n$.

Example 2.5 Solve $z^5 = 2i$.

$$z^5 = 2i = 2\exp\left(i\frac{\pi}{2} + 2i\pi n\right), \quad n = 0, 1, 2, \ldots$$
$$z = 2^{1/5}\exp\left(i\frac{\pi}{10} + \frac{2i\pi n}{5}\right).$$

Note that when $n = 5$ we repeat the root for $n = 0$, etc. So five roots are:

$$2^{1/5}\exp\left(i\frac{\pi}{10}\right), \quad 2^{1/5}\exp\left(i\frac{\pi}{2}\right) = 2^{1/5}i,$$
$$2^{1/5}\exp\left(i\frac{9\pi}{10}\right), \quad 2^{1/5}\exp\left(i\frac{13\pi}{10}\right) \quad 2^{1/5}\exp\left(i\frac{17\pi}{10}\right)$$

2.4 De Moivre's Theorem

We can use the exponential form of a complex number to derive a very useful result for obtaining trigonometric identities.

First, recall

$$e^{i\theta} = \cos\theta + i\sin\theta\ . \tag{116}$$

We can replace θ by $n\theta$ and write

$$e^{in\theta} = \cos n\theta + i\sin n\theta\ . \tag{117}$$

Also, we know that

$$e^{in\theta} = [e^{i\theta}]^n\ . \tag{118}$$

Combining these results yields

$$\boxed{\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n\ .} \tag{119}$$

This is called *De Moivre's Theorem*. Note that n does not have to be an integer.

De Moivre's Theorem, and the complex exponential more generally, are very useful indeed for working out expressions for multiple angle formulae, such as for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, and vice versa.

For instance, rewriting $\cos$ in terms of complex exponentials:

$$\begin{aligned} \cos^3\theta &= \left(\tfrac{1}{2}\right)^3 [\exp(i\theta)+\exp(-i\theta)]^3 \\ &= \tfrac{1}{8}[\exp(3i\theta)+3\exp(i\theta)+3\exp(-i\theta)+\exp(-3i\theta)] \\ &= \tfrac{1}{8}(2\cos 3\theta + 6\cos\theta) \\ &= \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta \ . \end{aligned} \tag{120}$$

Another application is to work out sums of trigonometric functions.

For example, if we wish to sum the series

$$\sum_{r=0}^{N} \cos r\theta \ ,$$

then the thing to do is to write

$$\sum_{r=0}^{N} \cos r\theta = \Re\left[\sum_{r=0}^{N} \exp(ir\theta)\right] \ . \tag{121}$$

The series

$$\sum_{r=0}^{N} \exp(ir\theta)$$

is actually a geometric progression with first term 1 and common ratio $\exp(i\theta)$, for which we can write down the answer, and then the cosine series we want follows by taking the real part. We do this in detail in Example 2.7.

<u>Example 2.6</u> Use De Moivre's Theorem to find expressions for $\cos 4\theta$ and $\sin 4\theta$.

$$\begin{aligned} \cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4, \\ = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ + 4i\cos^3\theta\sin\theta - 4i\cos\theta\sin^3\theta. \end{aligned}$$

Take real part of the equation:

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2, \\ &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

Take imaginary part:

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

Example 2.7 Evaluate $\sum_{k=0}^{N}\cos k\theta$.

First note that $\sum_{k=0}^{N}\cos k\theta = \mathrm{Re}\sum_{k=0}^{N}\exp(ik\theta)$. So what we have is a finite geometric sum: first term is 1, the common ratio is $\exp(i\theta)$. We can then use the formula for the geometric sum:

$$\begin{aligned}\sum_{k=0}^{N}\exp(ik\theta) &= \frac{1-\exp[i(N+1)\theta]}{1-\exp(i\theta)}, \\ &= \frac{1-\exp[i(N+1)\theta]}{1-\exp(i\theta)}\cdot\frac{1-\exp(-i\theta)}{1-\exp(-i\theta)}, \\ &= \frac{1-\exp(i\theta)-\exp[i(N+1)\theta]+\exp(iN\theta)}{2-2\cos\theta}.\end{aligned}$$

Take the real part:

$$\begin{aligned}\sum_{k=0}^{N}\cos(k\theta) &= \frac{1-\cos\theta-\cos(N+1)\theta+\cos N\theta}{2(1-\cos\theta)}, \\ &= \frac{1}{2}+\frac{\cos N\theta-\cos(N+1)\theta}{2(1-\cos\theta)}.\end{aligned}$$

2.5 Complex logarithms

Having discussed the complex exponential, the obvious thing to do next is to consider the inverse function, i.e. the complex natural logarithm $\ln z$. We first write z in the exponential form $z = |z|\exp(i\theta)$, and then:

$$\boxed{\begin{aligned} \ln z &= \ln(|z|\exp(i\theta)) \\ &= \ln(|z|) + \ln(\exp(i\theta)) \\ &= \ln(|z|) + i\theta \end{aligned}} \tag{122}$$

Of course, as we have already noted, the argument θ of z is really multi-valued, in the sense that we can add any integer multiple of 2π onto θ without changing the value of z. This means that $\ln z$ is a multi-valued function.

Often the *principal value* of $\ln z$ is defined by choosing just one of these possible values of θ, and the usual convention with $\ln z$ is to choose $-\pi < \theta < \pi$.

As an example, we work out $\ln(2i)$. First,

$$2i = 2\exp(i\pi/2 + 2n\pi i) \quad \text{for } n = 0, \pm 1, \pm 2, \dots\,, \tag{123}$$

and then using (122) we see that

$$\ln(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2\pi n\right) \ . \tag{124}$$

Example 2.8 Use complex logarithms to re-express 2^i and i^i.

$$2^i = [\exp(\ln 2)]^i = [\exp(\ln 2 + 2\pi i n)]^i = \exp(-2\pi n)\exp(i\ln 2), \qquad n = 0, 1, 2, \dots$$

Taking $n = 0$ gives us the principal value $\exp(i\ln 2)$.

$$i^i = [\exp(i\pi/2 + 2\pi i n)]^i = \exp(-\pi/2)\exp(-2\pi n), \qquad n = 0, 1, 2, \dots.$$

2.6 Oscillation problems

Complex numbers are especially useful in problems which involve oscillatory or periodic motion, such as when describing the motion of a simple pendulum, alternating electrical circuits, or any sort of wave motion in air and water.

To be specific, let us consider a simple pendulum swinging under gravity with angular frequency ω.

The angular displacement, $x(t)$, of the pendulum about the vertical then takes the general form

$$x(t) = a\cos\omega t + b\sin\omega t \; , \tag{125}$$

where a and b are real constants.

Using complex numbers we can write this as

$$x(t) = \Re\left[A\exp(i\omega t)\right] \; , \tag{126}$$

where A is a *complex* constant. In fact, by comparing (125) and (126) we find that

$$A = a - ib \; . \tag{127}$$

The big advantage of the complex representation (126) is that differentiation is very easy. For example, the velocity $v(t)$ is given by

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Re\left[A\exp(i\omega t)\right] = \Re\left[i\omega A\exp(i\omega t)\right] \; . \tag{128}$$

In other words, to differentiate we simply multiply by $i\omega$. This idea leads to various transform methods of solving differential equations.

3 Hyperbolic Functions

So far you've met a small number of important and commonly used functions, such as $\sin(x)$, $\cos(x)$, $\tan(x)$, $\exp(x)$, and $\ln(x)$. But there is, in fact, an entire zoo of interesting functions that emerge when solving various physical problems (Bessel functions, spherical harmonics, hypergeometric functions, elliptic integrals, etc.). In this section, we will introduce a class of new functions, that are related to the usual circular functions (cos, sin, tan). These are the *hyperbolic functions*.

3.1 Definitions

The hyperbolic functions are denoted and defined through:

$$\boxed{\begin{aligned} \cosh x &= \tfrac{1}{2}\left(e^{x}+e^{-x}\right), \\ \sinh x &= \tfrac{1}{2}\left(e^{x}-e^{-x}\right), \\ \tanh x &= \sinh x/\cosh x = \tfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}. \end{aligned}} \tag{129}$$

and are pronounced 'cosh', 'shine', and 'tansh'. In the same way as with circular trigonometric functions, we also define

$$\boxed{\begin{aligned} \operatorname{sech} x &= 1/\cosh x \\ \operatorname{cosech} x &= 1/\sinh x \\ \coth x &= 1/\tanh x\ . \end{aligned}} \tag{130}$$

There is in fact a very close relationship between circular and hyperbolic trigonometric functions that involves complex numbers. Recall equation (109),

$$\cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)\ .$$

Now consider

$$\cos(iz) = \frac{1}{2}\left[\exp(i^{2}z)+\exp(-i^{2}z)\right] = \frac{1}{2}\left(e^{-z}+e^{z}\right)\ ,$$

and comparing this result to equation (129) tells us that

$$\boxed{\cos(iz) = \cosh z\ .} \tag{131}$$

Similarly, recalling equation (110) we see that

$$\begin{aligned} \sin iz = \frac{1}{2i}\left[\exp(i^2 z) - \exp(-i^2 z]\right) &= \frac{1}{2i}\left(e^{-z} - e^{z}\right) \\ &= \frac{i}{2}\left(-e^{-z} + e^{z}\right)\ , \end{aligned}$$

so that

$$\boxed{\sin(iz) = i\sinh z\ .} \tag{132}$$

Dividing (132) by (131) gives

$$\boxed{\tan(iz) = i\tanh z\ .} \tag{133}$$

3.2 Identities

All the identities we know for circular trigonometric functions can be converted to corresponding identities for hyperbolic functions.

For example, consider $\cos^2 x + \sin^2 x = 1$. For hyperbolic functions, take

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \frac{1}{4}\left[\exp(x) + \exp(-x)\right]^2 - \frac{1}{4}\left[\exp(x) - \exp(-x)\right]^2 \\ &= \frac{1}{4}\left[\exp(2x) + \exp(-2x) + 2\right] - \frac{1}{4}\left[\exp(2x) + \exp(-2x) - 2\right] \\ &= \frac{1}{4}(4)\ , \end{aligned}$$

so that

$$\boxed{\cosh^2 x - \sinh^2 x = 1}\ . \tag{134}$$

Note the different sign compared to the classical trig relationship!

To derive multi-angle formulas we exploit the relationships between 'cosh' and 'cos' etc. For instance, start with the trigonometric identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \ .$$

This identity is actually true for all *complex* A and B, not just real values, so we could equally well have

$$\cos(iA+iB) = \cos iA \cos iB - \sin iA \sin iB \ ,$$

and then using (131) & (132) we find that

$$\cos(iA+iB) = \cosh(A+B) = \cosh A \cosh B - i^2 \sinh A \sinh B \ ,$$

so that

$$\boxed{\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B} \ . \tag{135}$$

Here are some more identities:

$$\boxed{\begin{array}{rcl} \sinh(A+B) &=& \sinh A \cosh B + \sinh B \cosh A \\ \sinh(A-B) &=& \sinh A \cosh B - \sinh B \cosh A \\ \cosh(A-B) &=& \cosh A \cosh B - \sinh A \sinh B \\ 1 - \tanh^2 z &=& \mathrm{sech}^2 z \\ \coth^2 z - 1 &=& \mathrm{cosech}^2 z \end{array} \ .}$$

<u>Example 3.1</u> Find an identity for $\tanh(A+B)$.

$$
\begin{aligned}
\tanh(A+B) &= \frac{\sinh(A+B)}{\cosh(A+B)}, \\
&= \frac{\sin(iA+iB)}{i\cos(iA+iB)}, \\
&= \frac{\sin(iA)\cos(iB)+\cos(iA)\sin(iB)}{i[\cos(iA)\cos(iB)-\sin(iA)\sin(iB)]}, \\
&= \frac{i\sinh A\cosh B + i\cosh A\sinh B}{i(\cosh A\cosh B - i^2\sinh A\sinh B)}, \\
&= \frac{\sinh A\cosh B+\cosh A\sinh B}{\cosh A\cosh B+\sinh A\sinh B}, \\
&= \frac{\tanh A+\tanh B}{1+\tanh A\tanh B},
\end{aligned}
$$

where in the last line we divided both numerator and denominator by $\cosh A\cosh B$.

3.3 Graphs of hyperbolic functions

When sketching the functions, note the following:

- $\cosh 0 = 1$, $\sinh 0 = \tanh 0 = 0$.
- The graph of $\cosh x$ is symmetric about $x = 0$, i.e. $\cosh x = \cosh(-x)$, while $\sinh x$ and $\tanh x$ are antisymmetric,
- As x approaches infinity through positive values, $y = \cosh x$ and $y = \sinh x$ approach the same curve $\frac{1}{2}\exp(x)$. This is because as x gets bigger $\exp(-x)$ is much less than $\exp(x)$,
- As x gets more and more negative, the term $\exp(-x)$ dominates. Hence $\sinh x$ approaches $-\frac{1}{2}\exp(-x)$, and $\cosh x$ approaches $\frac{1}{2}\exp(-x)$
- The previous two points explain why $\tanh x \to \pm 1$ as $x \to \pm\infty$.

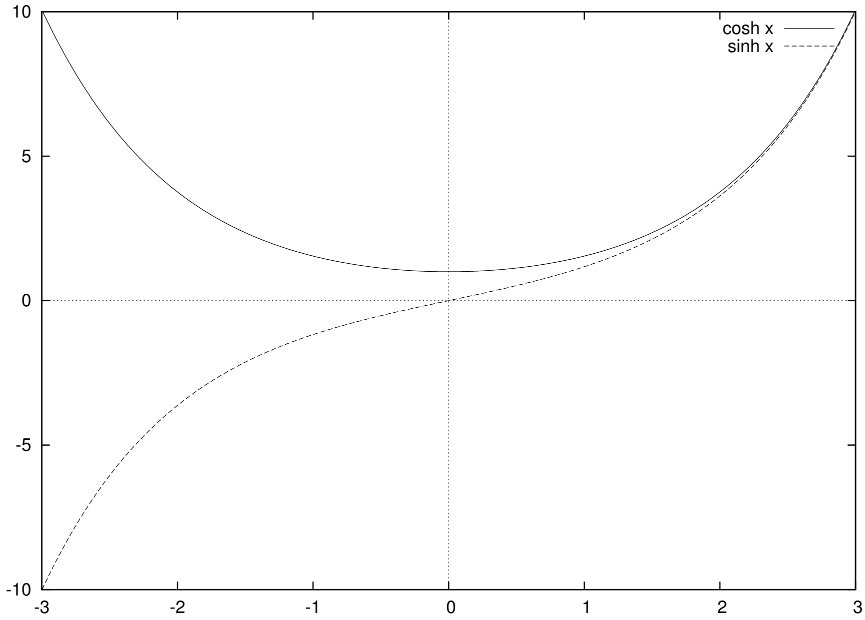

Figure 30: Plots of $\cosh x$ and $\sinh x$.

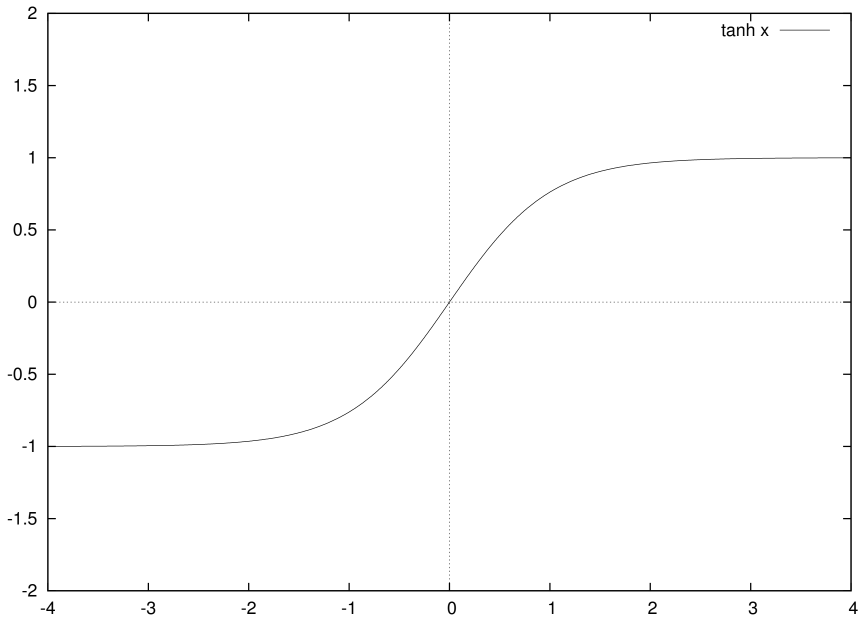

Figure 31: Plot of $\tanh x$. This is often used to represent a smooth step function.

- Note that $\cosh x \geqslant 1$ and $-1 < \tanh x < 1$ for all real x.

3.4 Inverse hyperbolic functions

Having defined the hyperbolic functions, we now want to introduce their inverses. Consider first the inverse of $\sinh x$, denoted

$$y = \sinh^{-1} x \ ,$$

which means

$$\sinh y = x \ . \tag{136}$$

Using the definition of sinh gives

$$\frac{1}{2}\left(e^y - e^{-y}\right) = x \ , \tag{137}$$

and multiplying through by $2e^y$ gives

$$e^{2y} - 2xe^y - 1 = 0 \ , \tag{138}$$

a quadratic equation in $\exp y$. The solutions are

$$e^y = x \pm \sqrt{x^2+1} \ . \tag{139}$$

Of these two roots, the one with the minus sign is negative, and must be thrown away (the exponential of a real number can never be negative).

Taking the log of the remaining root yields

$$\boxed{y = \sinh^{-1} x \equiv \ln\left(x + \sqrt{x^2+1}\right)} \ . \tag{140}$$

To obtain inverse cosh, set $y = \cosh^{-1} x$, and after almost the same algebra, we arrive at the equation

$$e^{2y} - 2xe^y + 1 = 0 \ , \tag{141}$$

with the two roots

$$e^y = x \pm \sqrt{x^2-1} \ . \tag{142}$$

Both roots are positive, so both must be kept.

Taking logs

$$y = \cosh^{-1} x \equiv \ln\left(x \pm \sqrt{x^2-1}\right) ,$$

our answer.

But the $\pm$ can be brought outside the log. First note that

$$x - \sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}} , \tag{143}$$

so that

$$\ln(x - \sqrt{x^2-1}) = \ln\left[\frac{1}{x+\sqrt{x^2-1}}\right] = -\ln(x + \sqrt{x^2-1}) .$$

We are therefore finally left with the answer

$$\boxed{y = \cosh^{-1} x \equiv \pm\ln\left(x + \sqrt{x^2-1}\right) .} \tag{144}$$

Obviously, $\cosh^{-1}$ is not defined when $x < 1$. That is because $\cosh(x) \geq 1$.

The reason that the inverse cosh is multivalued (i.e. possesses the $\pm$) is that cosh is symmetric ($\cosh x = \cosh(-x)$). So for every possible value of y on the $y = \cosh x$ curve there is a positive *and* a negative value of x.

Example 3.2 Find an identity for $\tanh^{-1} x$.

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

We now solve this equation for e^{2y}. Rearranging we get

$$x(e^{2y}+1) = e^{2y} - 1,$$
$$\to e^{2y}(x-1) = -x-1,$$
$$\to e^{2y} = \frac{1+x}{1-x},$$

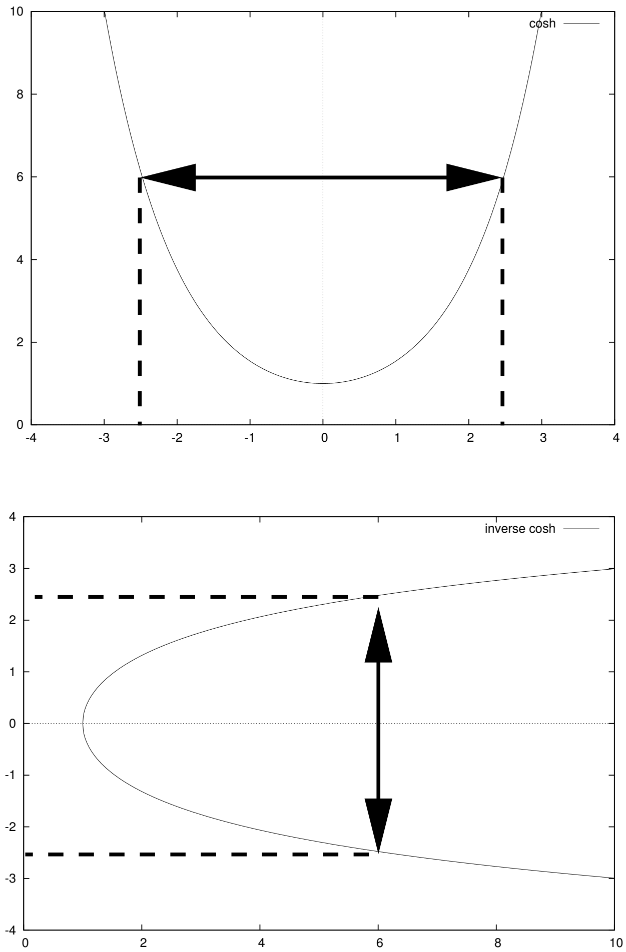

Figure 32: Plot of (a) $y = \cosh x$ and (b) $y = \cosh^{-1} x$.

and so finally:

$$y = \tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) .$$

Example 3.3 Find all the roots of $\cos z = 2$.

Obviously we are dealing with a complex root, because $\cos$ takes values between -1 and 1 for real arguments. Let $z = x + iy$, and so

$$\cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = 2,$$

$$\rightarrow \cos x \cosh y - i \sin x \sinh y = 2.$$

Take the real and imaginary parts

$$\cos x \cosh y = 2, \qquad -\sin x \sinh y = 0,$$

respectively. Let us solve the imaginary (second) part first. There are two possibilities:

1. $\sinh y = 0$, and so $y = 0$. Returning then to the real part of our original equation we have $\cos x = 2$, which is impossible because x has to be real. So this branch is no good.

2. $\sin x = 0$, and so $x = n\pi$, for $n = 0, 1, 2, \ldots$. Let us take n odd and go to the real part of the original equation, which becomes $-\cosh y = 2$. But then there are no real solutions for y (and y has to be real). So this is no good either.

   Let us finally take n even. The real part of the equation is now $\cosh y = 2$, which *can* be solved, and so $y = \cosh^{-1} 2 = \pm \ln(2 + \sqrt{3})$.

In summary there are an infinite number of complex solutions to the equation:

$$z = 2m\pi \pm i \ln(2 + \sqrt{3}), \qquad m = 0, 1, 2, \ldots$$

3.5 Circles, ellipses, and hyperbolae

3.5.1 Circles

As you all know, a circle in the xy plane with centre at the origin and with radius a has equation:

$$x^2 + y^2 = a^2. \tag{145}$$

But the curve can also be represented parametrically via

$$x = a\cos\theta, \qquad y = a\sin\theta, \tag{146}$$

where θ is the polar angle.

3.5.2 Ellipses

One way to generate an ellipse is to take a circle and then stretch one or both of the x and y axes. We then obtain the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{147}$$

where now a is called the *semi-major axis*, and b is the *semi-minor* axis. An ellipse need not have its semi-major and semi-minor axes aligned with the x and y axes, but in the canonical form above this is the case.

The curve has the parametric representation:

$$x = a\cos\theta, \qquad y = b\sin\theta, \tag{148}$$

where again θ is the polar angle.

An important quantity is the *eccentricity* $e = \sqrt{1 - b^2/a^2}$, which measures the degree of the ellipse's 'distortion'.

3.5.3 Hyperbolae

The equation of a hyperbola centred on the origin and aligned with the x and y axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{149}$$

Its parametric representation is

$$x = \pm a\cosh\theta, \qquad y = b\sinh\theta, \tag{150}$$

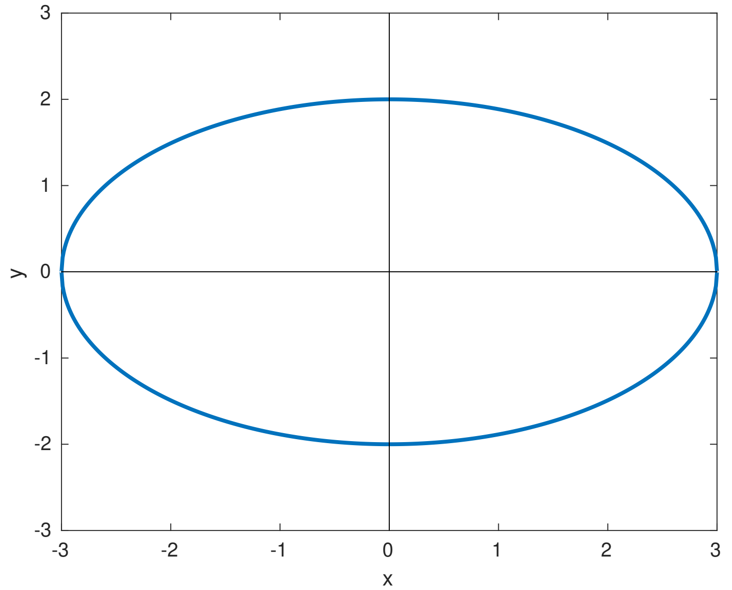

Figure 33: An ellipse with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.

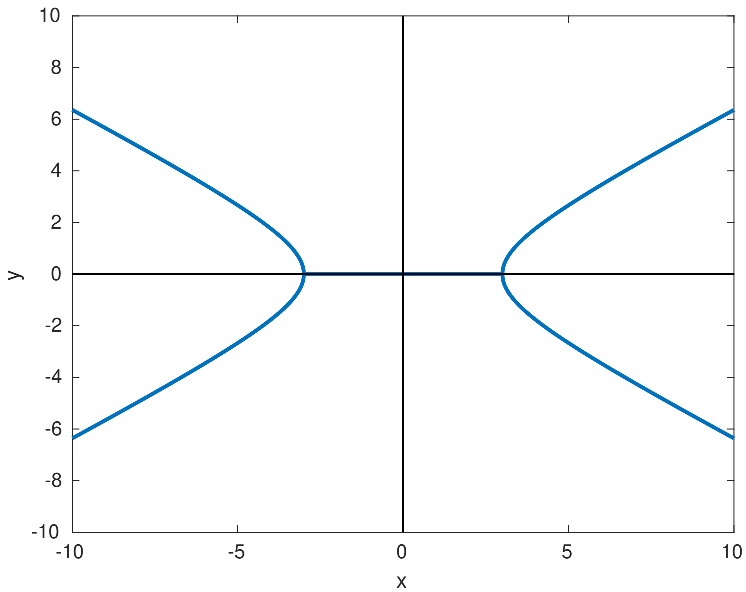

Figure 34: A hyperbola with semi-major axis of $a = 3$ and semi-minor axis of $b = 2$.