# Natural Sciences Tripos Part IA
# Mathematical Methods I — Course A
# Michaelmas 2020

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*"The book of nature cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics ... without which it is not humanly possible to understand a single word of it."* — **Galileo**

# Schedule

**Vectors**. Vector sum and vector equation of a line. Scalar product, unit vectors, vector equation of a plane. Vector product, vector area, vector and scalar triple products. Orthogonal bases. Cartesian components. Spherical and cylindrical polar coordinates. [5 lectures]

**Complex numbers**. Complex numbers and complex plane, vector diagrams. Exponential function of a complex variable. $\exp i\omega t$, complex representations of cos and sin. Hyperbolic functions. [3]

**Differential Calculus**. Revision for functions of a single variable of differentiation (including differentiation from first principles, product and chain rules) and of stationary values. Elementary curve sketching. Brief mention of the ellipse and its properties. Power series. Statement of Taylor's theorem. Examples to include the binomial expansion, exponential and trigonometric functions, and logarithm. Newton-Raphson method. [5]

**Integral calculus**. The integral as the limit of a sum. Methods of integration (including by parts and substitution). Examples to include odd and even functions and trigonometric functions. Fundamental theorem of calculus. [3]

**Probability**. Elementary probability theory. Simple examples of conditional probability. Probability distributions, discrete and continuous, normalisation. Permutations and combinations. Binomial distribution, $(p + q)^n$, binomial coefficients. Normal distribution. Expectation values, mean, variance and its expression in terms of first and second moments. [5]

**Examples**. Extended examples distributed through the course. [3]

**Course websites**

`www.natsci.tripos.cam.ac.uk/subject-information/part1a/maths`

`https://www.student-systems.admin.cam.ac.uk/moodle`
(search for NST Part IA: Mathematics)

`www.damtp.cam.ac.uk/user/hl278/NSTIA`

**Lecture notes and example sheets**

Lecture notes will be available online. They contain 'gaps' which will be filled-in during the lectures; periodically, *complete* lecture notes will be uploaded to Moodle. The online notes do not contain everything of value that will be given in the lectures, and it is expected that students will take down additional material during lectures to supplement them.

You will be given two examples sheets during Michaelmas term. These sheets will contain a mixture of basic 'skills' questions, and some more challenging questions. Brief solutions will be uploaded to Moodle towards the end of Michaelmas term and at the beginning of Lent term.

**Textbooks**

The following textbooks are recommended:

- M. L. Boas (1983). *Mathematical Methods in the Physical Sciences*, 3rd edition. Wiley.

- K. F. Riley, M. P. Hobson and S. J. Bence (2006). *Mathematical Methods for Physics and Engineering*, 3rd edition. Cambridge University Press

- E. Kreyszig (2005). *Advanced Engineering Mathematics*, 9th edition. Wiley.

- G. Stephenson (1973). *Mathematical Methods for Science Students*, 2nd edition. Prentice Hall/Pearson.

## Student representative

*"The Faculty Board of Mathematics asked DAMTP to set up a Staff-Student Committee for Mathematics in the Natural Sciences to provide an opportunity for discussion of matters relating to the courses. The Committee has four staff and three student members, the latter being drawn from the A and B courses in Part IA and from the Part IB course."*

A volunteer will be sought during October for this role. Please email me if interested.

## Questions and feedback

- Questions can be e-mailed to me: `hl278@cam.ac.uk`. Turnaround time: approximately 1-3 days

- Other routes for providing feedback or asking more general questions include the helpline `nst@maths.cam.ac.uk` and (once elected) your course representative.

# 1   Vectors

Vectors are indispensible in science. Systems in more than one dimension are most conveniently described using the formalism of vectors, while fluid dynamics and electromagnetism are inherently vector theories.

Vectors provide a compact notation that helps facilitate both calculations and physical understanding. Physical laws represented as vector equations are independent of the coordinate system; you can change your point of view (i.e. reference frame) but a vector equation does not change.

## 1.1   Definitions and properties

- **Scalars** are quantities with magnitude only
  *e.g. speed, mass, time, temperature.*

- **Vectors** have both magnitude *and* direction
  *e.g. displacement, velocity, force, momentum, magnetic field, electric field.*
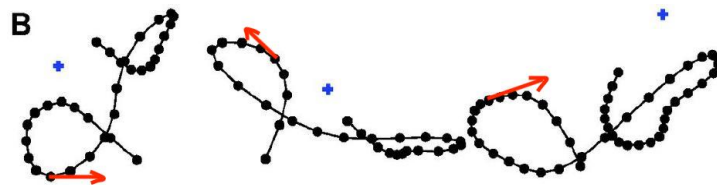


Figure 1: Three actual bee flights recorded by biologists. The instantaneous velocity vector of the bee at each point on the flight is tangent to each curve at that point (indicated by the red arrows).

- Geometrically, vectors are denoted with an arrow. The length of the arrow represents the magnitude of the vector.

- Algebraically, vectors are usually denoted by bold letters, i.e. $\boldsymbol{r}$, and sometimes as underlined letters, i.e. $\underline{r}$ (especially in handwriting). Then the mag-

nitude of the vector is written as $|\boldsymbol{r}|$.

- A vector of unit length is called a *unit vector*. Unit vectors are often denoted with a hat, i.e. $\hat{\boldsymbol{r}}$. And so $|\hat{\boldsymbol{r}}| = 1$.

- Constant vectors are not necessarily tied to a place in space; they are 'moveable' (cf. equipollent). Thus two vectors pointing in the same direction with the same magnitude *are the same*.
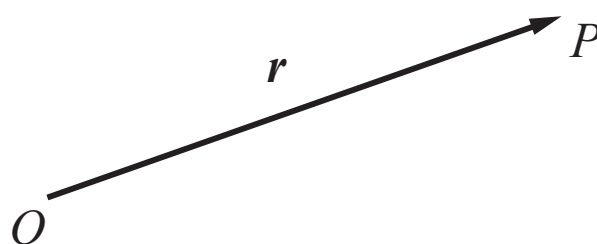
Figure 2: Displacement vector from the origin $O$ to the point $P$, that is, $\boldsymbol{r} = \overrightarrow{OP}$.

One particular sort of vector is the **displacement** vector between two points, say the displacement of the point $P$ from the point $O$. It is often denoted by the arrow over-bar notation $\overrightarrow{OP}$. We might also refer to $\overrightarrow{OP}$ as being the displacement vector of the point $P$ *relative to the origin* $O$, or as $P$'s *position vector*.

### 1.1.1   Multiplying vectors by scalars

A vector $\boldsymbol{a}$ can be multiplied by a scalar $\lambda$, to give another vector $\lambda\boldsymbol{a}$.

Figure 3: Two vectors: $\boldsymbol{a}$ (solid line) and $\lambda\boldsymbol{a}$ (dashed line)

- The vector $\lambda\boldsymbol{a}$ is simply the vector $\boldsymbol{a}$ but with its length scaled by $\lambda$.

- For $\lambda$ positive, the direction of the vector remains the same but the magnitude changes.

- Multiplying by a negative real scalar $\lambda$ reverses the direction of the vector.

- Two different vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, pointing in the *same* direction must be related by $\boldsymbol{a} = \mu\boldsymbol{b}$, where $\mu$ is some scalar.

### 1.1.2   Adding and subtracting vectors

We can also add and subtract vectors. It is easiest to define this operation geometrically.
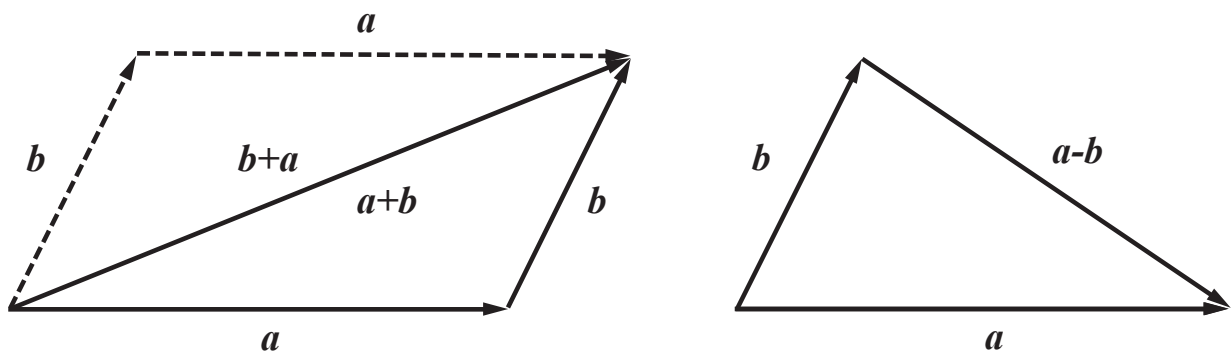


Figure 4: Adding and subtracting two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$.

- To add $\boldsymbol{a}$ and $\boldsymbol{b}$, arrange them so they form adjacent sides of a triangle, with arrow-tip to tail. Then $\boldsymbol{a} + \boldsymbol{b}$ is simply the third side of the triangle, with its arrow head meeting $\boldsymbol{b}$'s arrow head.

- If we construct a parallelogram from $\boldsymbol{a}$ and $\boldsymbol{b}$, we see that

$$\boxed{\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}} \, , \tag{1}$$

i.e. vector addition is **commutative**.

- The difference of two vectors, $a - b$, should be understood as $a + (-b)$. To compute it, first multiply $b$ by the scalar $-1$, then add to $a$ as normal.

- As well as being commutative, vector addition is also **associative**, i.e. the order in which the vectors are added does not matter;

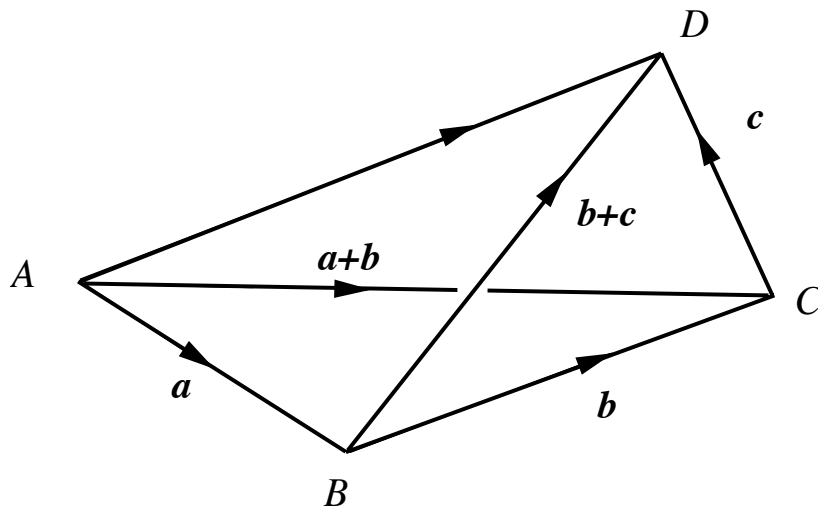$$\boxed{(a + b) + c = a + (b + c)} \ . \tag{2}$$



Figure 5: Diagram illustrating the associativity of vector addition.

- Finally, scalar multiplication and vector addition obey the two **distributive** laws; i.e.

$$\boxed{\lambda(a + b) = \lambda a + \lambda b} \tag{3}$$

$$\boxed{(\lambda + \mu)a = \lambda a + \mu a} \tag{4}$$

### 1.1.3 Cartesian coordinate axes

**Vector coordinates in two dimensions - simple case**

Consider the point $P$ in the plane, with Cartesian coordinates $(x, y)$.

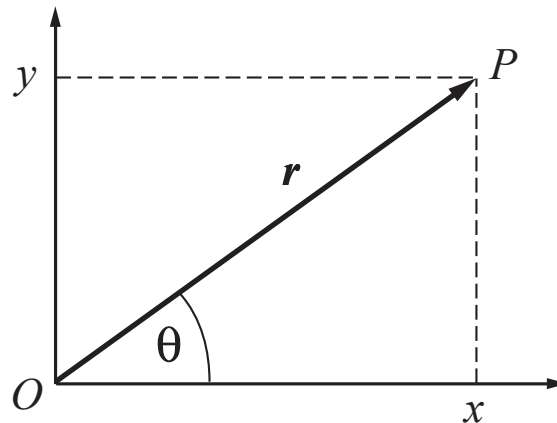Next consider the position vector of this point $r = \overrightarrow{OP}$, where $O$ is the origin.

Figure 6: position vector $\boldsymbol{r}$ in the plane with Cartesian axes.

- We can represent such position vectors by the coordinate pair $(x, y)$, and actually write $\boldsymbol{r} = (x, y)$

- We introduce two unit vectors, one along the $x$-axis, denoted by $\hat{\boldsymbol{i}}$ (or sometimes $\hat{\boldsymbol{x}}$), and one along the $y$-axis, denoted by $\hat{\boldsymbol{j}}$ (or $\hat{\boldsymbol{y}}$). These unit vectors are orthogonal (perpendicular) to each other. They may be represented via coordinates by

$$\hat{\boldsymbol{i}} = (1, 0) , \qquad \hat{\boldsymbol{j}} = (0, 1) . \tag{5}$$

- Via the laws of vector addition, our 2D vector $\overrightarrow{OP}$ can be represented as a *linear combination* of these unit vectors

$$\overrightarrow{OP} = \boldsymbol{r} = x\,\hat{\boldsymbol{i}} + y\,\hat{\boldsymbol{j}} , \tag{6}$$

and we see now that the constant coefficients of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are, in fact, also the coordinates of the point $P$.

- This representation permits us to do vector algebra component-wise: If $\boldsymbol{r}_1 =$

$(x_1, y_1)$ and $\boldsymbol{r}_2 = (x_2, y_2)$ are two position vectors,

$$\lambda \boldsymbol{r_1} = \lambda(x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}}) = (\lambda x_1, \lambda y_1), \tag{7}$$

$$\boldsymbol{r}_1 + \boldsymbol{r}_2 = x_1\hat{\boldsymbol{i}} + y_1\hat{\boldsymbol{j}} + (x_2\hat{\boldsymbol{i}} + y_2\hat{\boldsymbol{j}}) = (x_1 + x_2)\hat{\boldsymbol{i}} + (y_1 + y_2)\hat{\boldsymbol{j}},$$

$$= (x_1 + x_2, y_1 + y_2). \tag{8}$$

- From Pythagoras' Theorem, the *length* or *magnitude* of the vector $\boldsymbol{r}$ is

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2} \ . \tag{9}$$

- The *direction* of the 2D vector $\boldsymbol{r}$ can be represented by the unit vector $\hat{\boldsymbol{r}}$ lying in the same direction, that is,

$$\hat{\boldsymbol{r}} \ = \ \left(\frac{x}{r}, \frac{y}{r}\right) \ = \ \frac{x}{r}\,\hat{\boldsymbol{i}} + \frac{y}{r}\,\hat{\boldsymbol{j}} \ . \tag{10}$$

Alternatively we could also represent the direction using the angle $\theta$ subtended with the $x$-axis, where

$$\hat{\boldsymbol{r}} = (\cos\theta, \ \sin\theta) = \cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}} \ . \tag{11}$$

**Vector coordinates in three dimensions - general case**

- The *coordinates* of a point $P$ relative to three-dimensional Cartesian axes are denoted $x, y, z$ and correspond to the lengths of the projections of the vector $\overrightarrow{OP}$ onto the three axes.

- As before, we can write $\overrightarrow{OP} = (x, y, z)$.

- We introduce unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ which are parallel to the three axes. They are also commonly denoted by $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$, We have that

$$\hat{\boldsymbol{i}} = (1, 0, 0) \quad \hat{\boldsymbol{j}} = (0, 1, 0) \quad \hat{\boldsymbol{k}} = (0, 0, 1) \ , \tag{12}$$
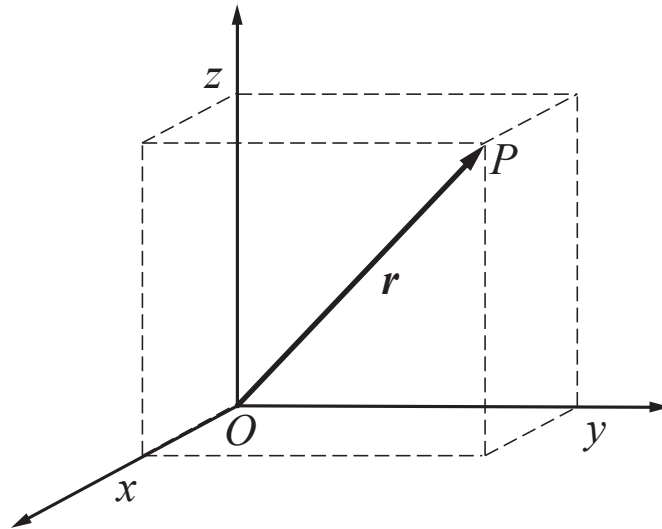
Figure 7: Position vector $r$ in 3D with Cartesian axes.

and then that

$$\overrightarrow{OP} = r = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \ . \tag{13}$$

The last equality follows from vector addition.

- The vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are said to form a *basis* for three-dimensional space, because any 3D vector can be written as a linear combination of these three vectors.

  Of course, this is not the only basis for three-dimensional space; we could obtain an infinite number of other bases by simply rotating $\hat{\imath}, \hat{\jmath}, \hat{k}$ by an arbitrary angle about the origin or by moving (displacing) the origin. We will come back to the idea of bases in section 1.11.

- As in 2D, vector algebra can be done component-wise in 3D. For example,

$$(x_1, y_1, z_1) \pm (x_2, y_2, z_2) = (x_1 \pm x_2, y_1 \pm y_2, z_1 \pm z_2).$$

- The *length* of the vector $r$ can be obtained from Pythagoras' Theorem and

is given by

$$\boxed{|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}} \, . \tag{14}$$

- The distance between the two points $\boldsymbol{r_1} = (x_1, y_1, z_1)$ and $\boldsymbol{r_2} = (x_1, y_1, z_1)$ can be found by noting first that

$$\boldsymbol{r_1} - \boldsymbol{r_2} = (x_1 - x_2, y_1 - y_2, z_1 - z_2) \, ,$$

so that from (14)

$$|\boldsymbol{r_1} - \boldsymbol{r_2}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \, . \tag{15}$$

Example 1.1 An aeroplane flies from city A to city B, which is at a distance of 1000km north-west of A. The average speed of the plane relative to the air immediately around it is $800\text{kmh}^{-1}$. For the journey in question, there is a wind of speed $80\text{kmh}^{-1}$ from the south-west. What direction would the pilot need to fly in, and how long does the journey take?

Let $\boldsymbol{v}_{\text{rel}}$ be the velocity of the plane relative to the air, and $\boldsymbol{v}_{\text{gr}}$ its velocity to the ground. We denote by $\overrightarrow{AB}$ the displacement vector between the two cities, and we want to set a course so that $\boldsymbol{v}_{\text{gr}}$ is parallel to $\overrightarrow{AB}$. Finally, $\boldsymbol{v}_{\text{wind}}$ is the wind velocity, perpendicular to $\overrightarrow{AB}$.

The plane's total velocity relative to the ground can be determine from vector addition (the plane propels itself but is also carried by the wind around it) so that

$$\boldsymbol{v}_{\text{gr}} = \boldsymbol{v}_{\text{rel}} + \boldsymbol{v}_{\text{wind}}.$$

The triangle so formed is a right angled triangle, so we can just use trigonometry to get the angle between $\boldsymbol{v}_{\text{rel}}$ and $\boldsymbol{v}_{\text{gr}}$, denoted $\phi$:

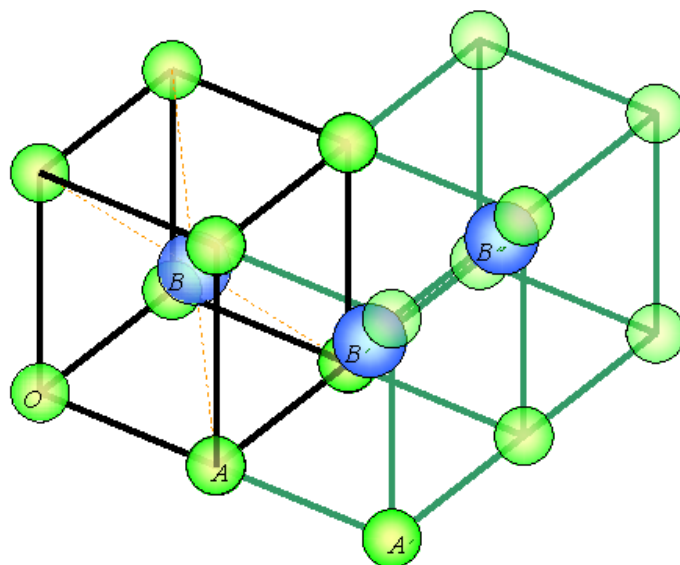$$\sin \phi = \frac{|\boldsymbol{v}_{\text{wind}}|}{|\boldsymbol{v}_{\text{gr}}|} = \frac{80}{800} = 0.1,$$

and so $\phi = \sin^{-1}(0.1) = 5.7°$.

To determine how long the flight takes we need to know $v_{gr}$, which follows from Pythagoras:

$$|v_{gr}| = \sqrt{|v_{rel}|^2 - |v_{wind}|^2} = \sqrt{800^2 - 80^2} = 80/\sqrt{99} \approx 796.0$$

The flight time is distance divided by speed $= 1000/796 = 1.2563$ hours.

(If there was no wind, the flight time is $1.25$ hours, and so the 80 km/h crosswind causes roughly 23 seconds delay!)



<u>Example 1.2</u> The crystalline lattice of caesium chloride consists of cubic unit cells. Each unit cell consists of a chlorine atom (labelled A) at each corner of the cube, and a caesium atom (labelled $B$) at the centre of the cube. The sides of the cube have length $a$. Find the distance from an atom $B$ to its nearest neighbouring atom of type (i) $A$ and (ii) $B$. What distances are the second nearest neighbours from $B$?

Let us set up a coordinate system aligned with the lattice and with origin $O$ placed at one of the $A$ atoms. There are hence $A$ atoms at $(0, 0, 0)$, $(a, 0, 0)$, $(0, a, 0)$. etc. And there are $B$ atoms at $(a/2, a/2, a/2)$, $(3a/2, a/2, a/2)$, $(3a/2, 3a/2, a/2)$, etc.

(i) Every $B$ atom has eight $A$ atoms surrounding it and these are equally close to it.

To get this distance choose the $A$ atom at the origin and the $B$ atom at $(a/2, a/2, a/2)$.

The length of the displacement vector is simply

$$|\boldsymbol{r}_A - \boldsymbol{r}_B| = |(0, 0, 0) - (a/2, a/2, a/2)| = (\sqrt{3}/2)a.$$

(ii) From the diagram the two nearest $B$ atoms are always located along the coordinate axes. Choosing $B$ and $B'$ in the diagram, the distance can be determined from the length of the displacement vector:

$$|\boldsymbol{r}_{B'} - \boldsymbol{r}_B| = \left\| \begin{pmatrix} \frac{3}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} - \begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} \right\| = a.$$

There are several $A$ atoms that are second nearest to $B$. Let us select $A'$ in the figure. Its position vector is $\boldsymbol{r}_{A'} = (2a, 0, 0)$. The distance between the two atoms can be determined from:

$$\begin{aligned}
|\boldsymbol{r}_{A'} - \boldsymbol{r}_B| &= \left\| \begin{pmatrix} 2a \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{pmatrix} \right\| = \left\| \begin{pmatrix} \frac{3}{2}a \\ -\frac{1}{2}a \\ -\frac{1}{2}a \end{pmatrix} \right\| \\
&= \sqrt{(\tfrac{3}{2}a)^2 + (-\tfrac{1}{2}a)^2 + (-\tfrac{1}{2}a)^2} \\
&= \frac{\sqrt{11}}{2}a.
\end{aligned}$$

The second-nearest $B$ atom from $B$ we take for our calculation to be $B''$ with

position vector $\boldsymbol{r}_{B''} = (3a/2, 3a/2, a/2)$. We quickly find that

$$|\boldsymbol{r}_{B''} - \boldsymbol{r}_B| = \sqrt{2}a.$$

Example 1.3 Recall that the cosine rule for the triangle illustrated below (Figure 8) is

$$c^2 = a^2 + b^2 - 2ab\cos\gamma, \tag{16}$$

where $a = |\boldsymbol{a}| = |\overrightarrow{OA}|$, $b = |\boldsymbol{b}| = |\overrightarrow{OB}|$ and $c = |\boldsymbol{c}| = |\overrightarrow{BA}|$. The sine rule here is

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}. \tag{17}$$
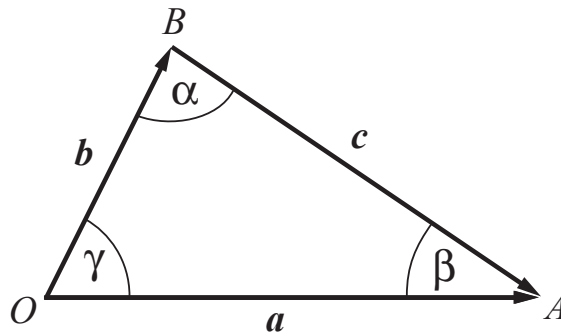


Figure 8: Vector and angle definitions for the cosine and sine rules.

Use the cosine rule to calculate the angle between the lines OA and OB, where O is the origin, $A = (2, -1, 2)$ and $B = (1, 1, 1)$.

We first recognise that the two lines make a triangle once we join the two points $A$ and $B$.

We next re-arrange the cosine rule to isolate $\gamma$ the angle between the two lines and the thing we want to find:

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

Next we need to find all the lengths $a$, $b$, and $c$:

$$a = |\boldsymbol{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3,$$

$$b = |\boldsymbol{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3},$$

$$c = |\boldsymbol{b} - \boldsymbol{a}| = \sqrt{(1-2)^2 + (1+1)^2 + (1-2)^2} = \sqrt{6},$$

where we have used vector addition in the last line.

We then just plug the numbers into the formula:

$$\cos\gamma = \frac{9 + 3 - 6}{2 \times 3 \times \sqrt{3}} = \frac{1}{\sqrt{3}},$$

and so $\gamma = \cos^{-1}\sqrt{1/3} = 54.7°$.

## 1.2   The equation of a line

Consider a line in the plane that passes through two points $A$ and $B$, with $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. How can we describe a general point $P$ that lies on this line? What equation does its position vector $\boldsymbol{r}$ satisfy?

Note that the line between $A$ and $P$ is necessarily parallel to the line between $A$ and $B$. This means that $\overrightarrow{AP}$ must be a scalar multiple of $\overrightarrow{AB}$, or in other words

$$\boldsymbol{r} - \boldsymbol{a} = \lambda(\boldsymbol{b} - \boldsymbol{a}),$$

for some scalar $\lambda$. Rearranging yields

$$\boxed{\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})}. \tag{18}$$

Equation (18) is the equation of a straight line. Any point $\boldsymbol{r}$ on the line can be written in the form (18) for some $\lambda$. The scalar $\lambda$ runs through all real values.

Instead of knowing two points on the line, we are sometimes given just one point (say $\boldsymbol{a}$) together with the direction of the line (the unit vector $\hat{\boldsymbol{t}}$).
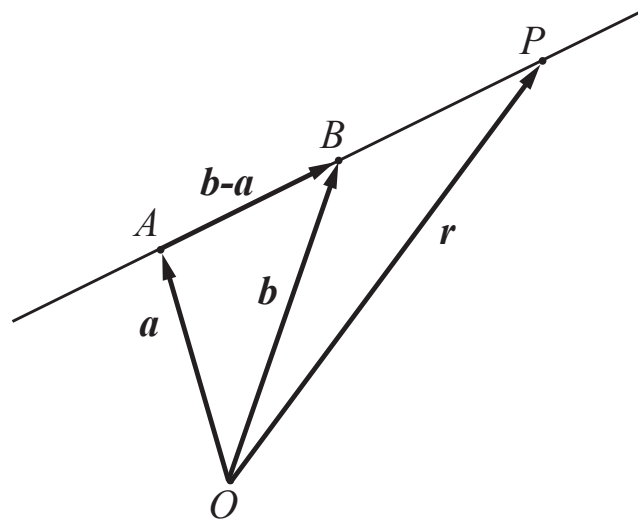
Figure 9: A straight line that goes through points $A$ and $B$, the origin $O$, and a random point on the line $P$.

In that case, the equation of the line can easily be modified to give the alternate general (and more convenient) form

$$\boxed{r = a + \lambda \hat{t}} \, . \tag{19}$$

### 1.2.1   Component description - two dimensions

To keep things simple to start, suppose the line lies in the $xy$ plane.

Now a point on the line can be written as $r = (x, y)$. And the equation of the line, in components, is

$$(x, y) = (a_x + \lambda(b_x - a_x), \, a_y + \lambda(b_y - a_y)), \tag{20}$$

where $a = (a_x, a_y)$ and $b = (b_x, b_y)$.

This implies both $\lambda = (x - a_x)/(b_x - a_x)$ and $\lambda = (y - a_y)/(b_y - a_y)$.

Eliminating $\lambda$ yields the familiar:

$$\frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} \qquad \text{or} \qquad y = c \, x + d \, , \tag{21}$$

where the slope $c = (b_y - a_y)/(b_x - a_x)$ and the intercept $d = a_y - c \, a_x$.

### 1.2.2   Component description - three dimensions

In three dimensions, the component form of the line is a bit more involved.

Let now $r = (x, y, z)$. We once again rewrite the equation of a line in Cartesian components:

$$(x, y, z) = (a_x + \lambda(b_x - a_x), a_y + \lambda(b_y - a_y), a_z + \lambda(b_z - a_z)),$$

where $a = (a_x, a_y, a_z)$ and $b = (b_x, b_y, b_z)$.

Equating the components yields three equations

$$\lambda = \frac{x - a_x}{b_x - a_x} \quad, \quad \lambda = \frac{y - a_y}{b_y - a_y} \quad, \quad \lambda = \frac{z - a_z}{b_z - a_z}.$$

Of course $\lambda$ is the same in each of these three equations, so the right hand sides must all be equal to each other.

On eliminating $\lambda$, we obtain

$$\boxed{\frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} = \frac{z - a_z}{b_z - a_z}}, \tag{22}$$

which is an alternative form of (18) not involving the scalar parameter $\lambda$.

Example 1.4 Find the equation of the straight line through the points $a = (1, 2, 3)$, $b = (4, 0, 6)$. Does this line pass through the point $(4, 0, 0)$? For what value of $\mu$ does the point $(10, -4, \mu)$ lie on the line?

Plug in our information into the line equation:

$$r = a + \lambda(b - a),$$
$$= (1, 2, 3) + \lambda[(4, 0, 6) - (1, 2, 3)]$$
$$= (1 + 3\lambda, 2 - 2\lambda, 3 + 3\lambda).$$

Now is $r = (4, 0, 0)$ on this line? Plug it in and see if the line equation can be satisfied for some $\lambda$.

The three components of the equation give

$$4 = 1 + 3\lambda, \quad 0 = 2 - 2\lambda, \quad , 0 = 3 + 3\lambda.$$

The first equations yields $\lambda = 1$ but the third equation demands that $\lambda = -1$. So we have a contradiction, which can only be resolved if the point $(4, 0, 0)$ does not lie on the line.

We play the same game with $\boldsymbol{r} = (10, -4, \mu)$. We plug in this expression into the line equation. The $x$ component gives:

$$10 = 1 + 3\lambda \qquad \rightarrow \lambda = 3,$$

while the $y$ component yields $-4 = 2 - 2\lambda$, which is consistent with $\lambda = 3$. Finally the $z$ component is:

$$\mu = 3 + 3\lambda = 12.$$

So the point lies on the line when $\mu = 12$.

Example 1.5 Consider the triangle $ABC$, with the position vectors of the points relative to the origin $O$ being $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$ respectively. Show that the medians (the lines from each vertex to the middle of the opposite side) meet in a single point, the so-called centroid.

Let $D$ be the mid point on the line $BC$, and let its position vector be $\boldsymbol{d} = \overrightarrow{OD}$. Consider the triangle $OBD$; from vector addition,

$$\boldsymbol{d} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \boldsymbol{b} + \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}).$$

Next let us find the equation of the line joining points $A$ and $D$ , i.e. the median joining the vertex $A$ and the midpoint of $BC$. This is simply:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{d} - \boldsymbol{a}),$$
$$= \boldsymbol{a} + \lambda\left[\tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{a}\right],$$
$$= (1 - \lambda)\boldsymbol{a} + \tfrac{1}{2}\lambda\boldsymbol{b} + \tfrac{1}{2}\lambda\boldsymbol{c},$$

where $\lambda$ is some real scalar parameter.

If we repeat this process, we can find the equation of the median joining the vertex $B$ and the midpoint of $AC$:

$$r = (1 - \mu)b + \tfrac{1}{2}\mu a + \tfrac{1}{2}\mu c,$$

for some other real parameter $\mu$. And indeed the equation of the median joining $C$ to the midpoint of $AB$ is:

$$r = (1 - \nu)c + \tfrac{1}{2}\nu a + \tfrac{1}{2}\nu b,$$

for a parameter $\nu$.

Now we are in a position to judge where these lines meet and if they do meet at the same point. Let us see where the first two lines intersect, by setting

$$(1 - \lambda)a + \tfrac{1}{2}\lambda b + \tfrac{1}{2}\lambda c = (1 - \mu)b + \tfrac{1}{2}\mu a + \tfrac{1}{2}\mu c.$$

Equating coefficients of $a$, $b$, and $c$ gives us three equations. If we look at $c$ first we find $\lambda = \mu$. Then turning to both $a$ and $b$ we get the same equation: $1 - \lambda = \lambda/2$. And thus

$$\lambda = \mu = \frac{2}{3}.$$

The position vector of the point is then:

$$r = \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c.$$

Finally, we need to check that this point lies on the third line. If we plug it in to the equation we have:

$$\tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c = (1 - \nu)c + \tfrac{1}{2}\nu a + \tfrac{1}{2}\nu b,$$

which can be satisfied when $\nu = 2/3$.

So all three medians intersect at the same point.

## 1.3  Scalar product

We consider two different ways of multiplying vectors together. The first is the **scalar product**, in which two vectors are multiplied together to give a scalar.

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, whose directions have an angle $\theta$ between them (without loss of generality $0 \leqslant \theta \leqslant \pi$).

The *scalar*, or *dot*, product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by the formula

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta} \ . \tag{23}$$
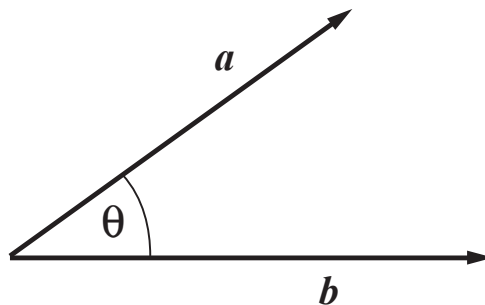


Figure 10: Angle used in the definition of the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$

Scalar product properties:

1. The scalar product is commutative,

   $$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a} \ , \tag{24}$$

   which follows straight from (23).

2. The scalar product is distributive,

   $$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \ , \tag{25}$$

   a fact which is most easily proven using the component formula given below.

3. The scalar product of a vector with itself is denoted by convention as the square of the vector, i.e. $\boldsymbol{a}^2 \equiv \boldsymbol{a} \cdot \boldsymbol{a}$. It follows from (23) that

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 \,,$$

i.e. the length of $\boldsymbol{a}$ squared. Using (25) and then (24) yields the useful result

$$(\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a} \cdot \boldsymbol{b} \,. \tag{26}$$

4. If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, and $|\boldsymbol{a}|, |\boldsymbol{b}| \neq 0$ then $\cos\theta = 0$, necessarily, i.e. the vectors are orthogonal.

5. The unit vectors $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ have the following properties:

$$\begin{aligned}
\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} &= \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} &= \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} &= 1 \,, \\
\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} &= \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} &= \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}} &= 0
\end{aligned} \tag{27}$$

In fact, $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ are often referred to as being an *orthonormal* basis set; 'ortho-' meaning orthogonal to each other and 'normal' meaning that each has length unity.

### 1.3.1 Component formula

We now give an expression for the scalar product of two vectors $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$ which are given in component form. We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}) \cdot (b_x\hat{\boldsymbol{i}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}}) \,.$$

We use the distributive property (25) to expand the brackets, and then the commutative property (24) to see that $\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{i}}$ etc. This produces

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} &+ a_y b_y \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} + a_z b_z \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} + (a_x b_y + a_y b_x)\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} \\
&+ (a_y b_z + a_z b_y)\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} + (a_z b_x + a_x b_z)\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{i}} \,,
\end{aligned}$$

and finally using the orthonormality in (27) gives us

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z} \, . \tag{28}$$

*Exercise:* Use the property of the scalar product (26) to prove the cosine rule (16). [Hint: See the triangle in Figure 8 with $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$.]

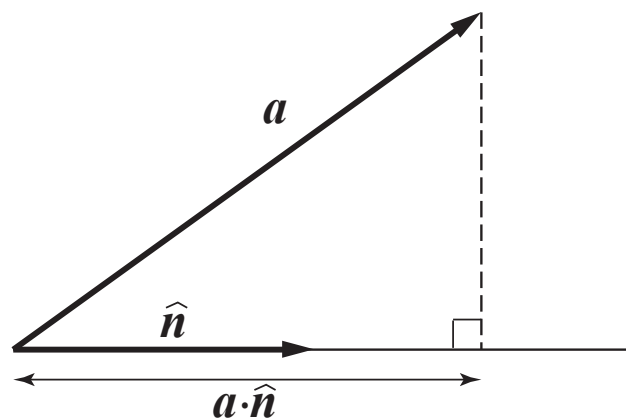### 1.3.2   Component and projection of a vector in a given direction



Figure 11: The component of a vector $\boldsymbol{a}$ in the direction of the unit vector $\hat{\boldsymbol{n}}$.

Often we need to work out the component of a vector $\boldsymbol{a}$ in a given direction, say in the direction of the unit vector $\hat{\boldsymbol{n}}$.

By simple trigonometry, this component is $|\boldsymbol{a}| \cos\theta$, where $\theta$ is the angle between $\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$.

But this can be rewritten using the scalar product: the component of the vector $\boldsymbol{a}$ in the direction of the unit vector $\hat{\boldsymbol{n}}$ is just $\boldsymbol{a} \cdot \hat{\boldsymbol{n}}$.

The **projection** of $\boldsymbol{a}$ in the direction $\hat{\boldsymbol{n}}$ is a vector and is given by $(\boldsymbol{a} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}$. It is sometimes also referred to as the 'vector resolution' in $\hat{\boldsymbol{n}}$.

<u>Example 1.6</u> Find the angle between the vectors $(1, 2, 2)$ and $(0, 3, -4)$. Find the component of the vector $(2, -1, 7)$ in the direction $(1, 1, 1)$.

Call the first vector $a$ and the second $b$. To get the angle $\theta$ between them use the dot product.

We know using coordinates that

$$a \cdot b = (1, 2, 2) \cdot (0, 3, -4) = 1 \times 0 + 2 \times 3 + 2 \times (-4) = -2.$$

But it also equals $a \cdot b = |a||b| \cos \theta$. The amplitudes are easy enough to work out

$$|a| = \sqrt{1 + 2^2 + 2^2} = \sqrt{9} = 3, \quad |b| = \sqrt{0 + 3^2 + (-4)^2} = 5.$$

We then equate the two expressions for the dot product and solve for $\theta$:

$$-2 = 15 \cos \theta, \qquad \rightarrow \theta = \cos^{-1}\left(\frac{-2}{15}\right) = 97.7°.$$

The component of the vector $c = (2, -1, 7)$ in the direction $n = (1, 1, 1)$ is just $c \cdot \hat{n}$. The magnitude of $n$ is just $\sqrt{3}$, so

$$c \cdot \hat{n} = (2, -1, 7) \cdot \frac{1}{\sqrt{3}}(1, 1, 1) = \frac{8}{\sqrt{3}}.$$

## 1.4   The equation of a plane

A general plane in three dimensions can be specified by

1. The orientation of the plane, given by the direction of the unit normal vector to the plane $\hat{n}$

2. One point on the plane, with position vector $a$, say.

Take the general point $P$ on the plane to have position vector $r$. The vector $r - a$ must lie in the plane, and thus is perpendicular to the normal to the plane $\hat{n}$, and therefore by property (4) of the scalar product

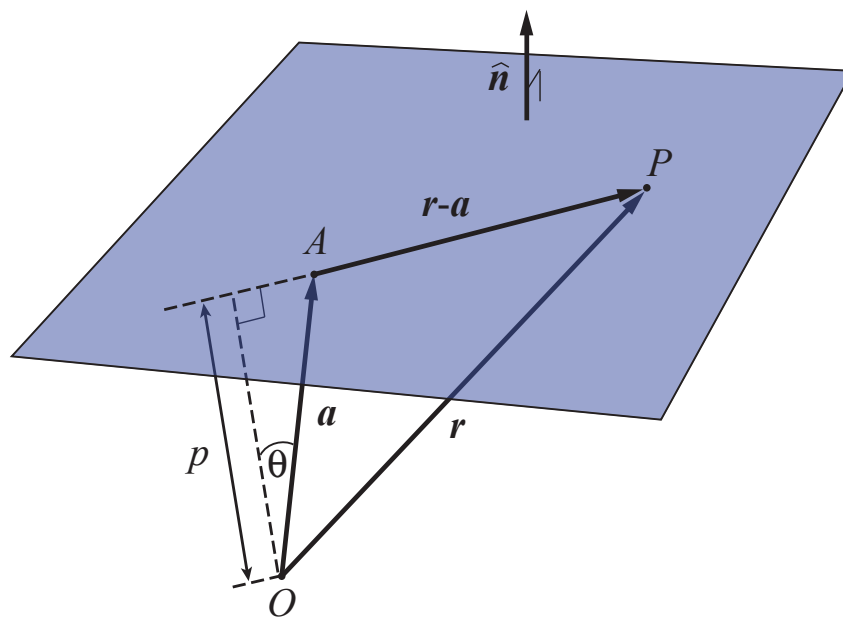$$\boxed{(r - a) \cdot \hat{n} = 0}, \tag{29}$$

Figure 12: A plane defined by the unit normal vector $\hat{\boldsymbol{n}}$ and the point $A$.

which is one form of the general equation of a plane.

One important property of the plane is its perpendicular distance $p$ to the origin $O$. From trigonometry

$$p = |\boldsymbol{a}| \cos \theta \, ,$$

but since $\theta$ is the angle between $\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$ we have that $p = \boldsymbol{a} \cdot \hat{\boldsymbol{n}}$.

In fact, this yields another form of the equation of the plane,

$$\boxed{\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = p} \, . \tag{30}$$

### 1.4.1 Component formula

We can also write the equation of the plane in component form. Suppose that $\boldsymbol{r} = (x, y, z)$ and $\hat{\boldsymbol{n}} = (l, m, n)$, then the equation of the plane is simply

$$\boxed{lx + my + nz = p} \, . \tag{31}$$

The real numbers $l, m, n$ are called the *direction cosines*, because they correspond to the cosine of the angle between the normal $\hat{\boldsymbol{n}}$ and the $x, y, z$ axes respectively.

For instance $l = \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}} = \cos\vartheta$, where $\vartheta$ is the angle between $\hat{\boldsymbol{n}}$ and the $x$-axis, i.e. $\hat{\boldsymbol{i}}$. Similarly, $m$ and $n$ are the cosines of the angles made by $\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ respectively.

### 1.4.2   Alternative equation of the plane (if given three points)
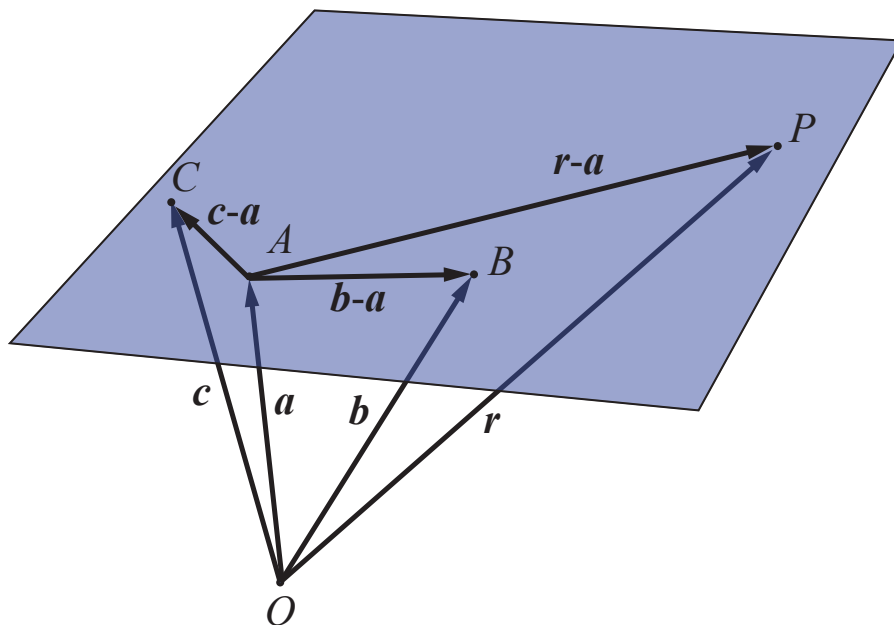


Figure 13:  A plane containing the points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and the general point $\boldsymbol{r}$.

We can also find the equation of the plane if we know three points in the plane.

- Suppose that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the position vectors of three points $A$, $B$ and $C$ in the plane.

- Then the vectors $\boldsymbol{c} - \boldsymbol{a}$ and $\boldsymbol{b} - \boldsymbol{a}$ both lie entirely within the plane.

- The plane itself is two-dimensional, and that means that any other vector which lies in the plane can be made as a linear combination of $\boldsymbol{c} - \boldsymbol{a}$ and $\boldsymbol{b} - \boldsymbol{a}$.

- In particular, the vector $\boldsymbol{r} - \boldsymbol{a}$ lies within the plane, so $\boldsymbol{r} - \boldsymbol{a}$ must be the sum of a scalar multiple of $\boldsymbol{b} - \boldsymbol{a}$ plus a scalar multiple of $\boldsymbol{c} - \boldsymbol{a}$, i.e.

$$\boxed{\boldsymbol{r} - \boldsymbol{a} = \mu(\boldsymbol{b} - \boldsymbol{a}) + \nu(\boldsymbol{c} - \boldsymbol{a})} . \tag{32}$$

Equation (32) is yet another form for the equation of a plane, and the idea is that the scalars $\mu, \nu$ range over all real values so as to describe all possible points on the plane; we clearly need two degrees of freedom, $\mu, \nu$, to describe a two-dimensional object.

Note that in deriving (32) we have assumed that $\boldsymbol{c} - \boldsymbol{a}$ and $\boldsymbol{b} - \boldsymbol{a}$ are *not* parallel to each other.

Example 1.7 Find the line of intersection of the two planes with normals parallel to the vectors $(1, 0, 2)$ and $(-1, 1, 1)$ which both pass through the point $(1, 1, 0)$. What angles does this line make with the coordinate axes?

The equation of the plane may be written as $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$ where $\boldsymbol{a}$ is a point on the plane and $\boldsymbol{n}$ is a vector perpendicular to the plane. Note that $\boldsymbol{n}$ need not be a unit vector for this equation to hold (though it is sometimes convenient to frame things this way).

The equation of the first plane can then be written:

$$[(x, y, z) - (1, 1, 0)] \cdot (1, 0, 2) = 0, \qquad \rightarrow \qquad x - 1 + 2z = 0.$$

The equation of the second plane is:

$$[(x, y, z) - (1, 1, 0)] \cdot (-1, 1, 1) = 0, \qquad \rightarrow \qquad x - y - z = 0.$$

Intersection happens on points that satisfy both equations. We deduct the second equation from the first and get $y + 3z = 1$, and so $z = (1 - y)/3,$

putting this in to the first plane's equation yields $z = (1 - x)/2$. Thus the equations of the line may be written

$$\frac{x - 1}{-2} = \frac{y - 1}{-3} = z.$$

Comparing with Eq.(22) a vector parallel to the line can be determined by looking at the denominators. Call it $t = (-2, -3, 1)$. Its magnitude is $\sqrt{4 + 9 + 1} = \sqrt{14}$. Therefore

$$\hat{t} = \left( \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right).$$

However the components of $\hat{t}$ are also the direction cosines with respect to three coordinate axes, i.e. $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$. So it's easy to compute these angles:

$$\theta_x = \cos^{-1}\left( \frac{-2}{\sqrt{14}} \right) = 122°,$$

$$\theta_y = \cos^{-1}\left( \frac{-3}{\sqrt{14}} \right) = 143°,$$

$$\theta_z = \cos^{-1}\left( \frac{1}{\sqrt{14}} \right) = 74°.$$

Example 1.8 Can an object at the point $(-2, 3, 3/2)$, be seen by an observer $B$ at $(4, 5, 1)$ when there is an intervening wall with top given by the line

$$r = (1, 2, 1) + \lambda(1, -1, -1)$$

[the $z$ axis is the upward vertical]?

The observer and the line along the top of the wall define a plane in 3D space. Objects above this plane (with respect to $z$) can be seen by the observer and objects below this plane cannot be seen. Our job is to determine the equation of the plane and then its height $z$ at the $xy$ location of the object.

Equation of the plane. We know two points on the plane $A$, given by $a = (1, 2, 1)$ and $B$. We can find a third point $C$ by going to the line in the plane

and setting $\lambda = 1$ which gives $c = a + (1, -1, 1) \left[ = (2, 1, 0) \right]$. We can use the equation of the plane with these three points:

$$r - a = \mu(b - a) + \nu(c - a),$$

where $\mu$ and $\nu$ are real scalar parameters. We thus have

$$r = (1, 2, 1) + \mu(3, 3, 0) + \nu(1, -1, -1).$$

Equating components we have

$$x = 1 + 3\mu + \nu, \quad y = 2 + 3\mu - \nu, \quad z = 1 - \nu.$$

Now, the object is at $(-2, 3, 3/2)$. Putting in the horizontal position ($x = -2, y = 3$) in to the equation of the plane yields two equations for $\mu$ and $\nu$:

$$-2 = 1 + 3\mu + \nu, \qquad 3 = 2 + 3\mu - \nu,$$

which we solve simultaneously. Add the two equations together and we get $1 = 3 + 6\mu$, thus $\mu = -1/3$. And then $\nu = -2$. This means at this horizontal location the height of the plane is

$$z = 1 - \nu = 3,$$

which is *above* the height of the object ($z = 3/2$), and so the object *cannot* be viewed by the observer!

## 1.5   Equations of other objects

### 1.5.1   Sphere

Consider a sphere of radius $R$ and centre whose position vector is $a$.

A point $r$ on the surface of the sphere is, by definition, a distance $R$ from the centre $a$, and it follows that the equation of the sphere is

$$\boxed{|r - a| = R} \,. \tag{33}$$

In component form with $\boldsymbol{a} = (a_x,\, a_y,\, a_z)$, we obtain the more familiar form of the equation for the sphere

$$(x - a_x)^2 + (y - a_y)^2 + (z - a_z)^2 = R^2 \,.$$

## 1.5.2  Cylinder
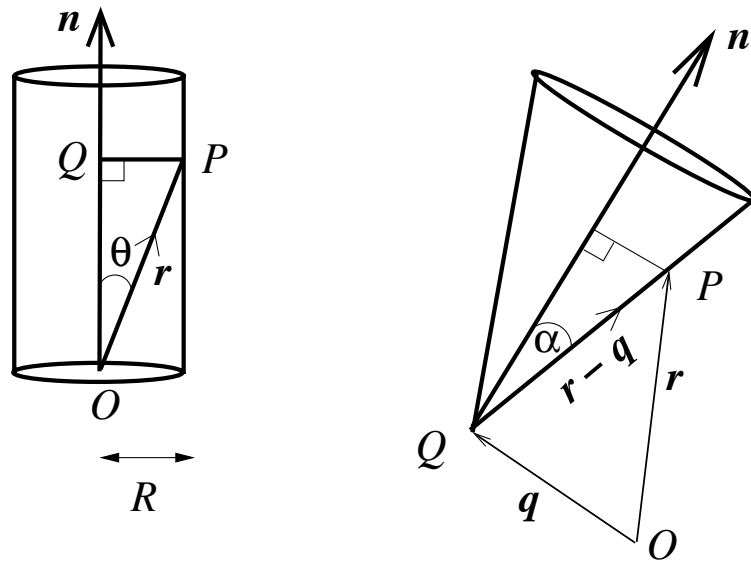


Figure 14: A cylinder and a cone.

Consider a circular cylinder of radius $R$ with axis parallel to the unit vector $\hat{\boldsymbol{n}}$. The origin $O$ lies on the cylinder axis.

- A general point on the surface $P$ has position vector $\overrightarrow{OP} = \boldsymbol{r}$.

- Suppose $\overrightarrow{OQ}$ is the projection of $\overrightarrow{OP}$ onto the cylinder's axis. The distance between $O$ and $Q$ is

$$|\overrightarrow{OQ}| = |\boldsymbol{r}| \cos \theta = |\boldsymbol{r} \cdot \hat{\boldsymbol{n}}| \tag{34}$$

Since $\overrightarrow{OQ}$ is parallel to the cylinder axis,

$$\overrightarrow{OQ} = (\boldsymbol{r} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}.$$

- We can also see that $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$, and that $|\overrightarrow{QP}| = R$ the cylinder radius. Putting all this information together leads us to the vector equation of the cylinder

$$\boxed{|\boldsymbol{r} - (\boldsymbol{r} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}}| = R} \, . \tag{35}$$

If the cylinder lies along the $z$-direction with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, then in components the equation is simply $x^2 + y^2 = R^2$.

### 1.5.3  Cone

Consider a cone with semi-angle $\alpha$, its axis parallel to the unit vector $\hat{\boldsymbol{n}}$, and its vertex at the point $Q$ with position vector $\boldsymbol{q}$.

- A general point on the cone surface is $P$ with $\overrightarrow{OP} = \boldsymbol{r}$.

- The defining property of the cone is that the line from the vertex to any point on the surface always makes an angle $\alpha$ with axis. Mathematically this means

$$\boxed{(\boldsymbol{r} - \boldsymbol{q}) \cdot \hat{\boldsymbol{n}} = |\boldsymbol{r} - \boldsymbol{q}| \cos \alpha} \, , \tag{36}$$

  which is the vector equation of the cone.

## 1.6  Vector product

We now introduce the *vector product* of two vectors, $\boldsymbol{a} \wedge \boldsymbol{b}$, which is itself a vector. This is also sometimes referred to as the *cross product*, and is sometimes given the notation $\boldsymbol{a} \times \boldsymbol{b}$.

The definition of the vector product is

$$\boxed{\boldsymbol{a} \wedge \boldsymbol{b} \equiv |\boldsymbol{a}||\boldsymbol{b}| \sin \theta \, \hat{\boldsymbol{n}}} \, , \tag{37}$$
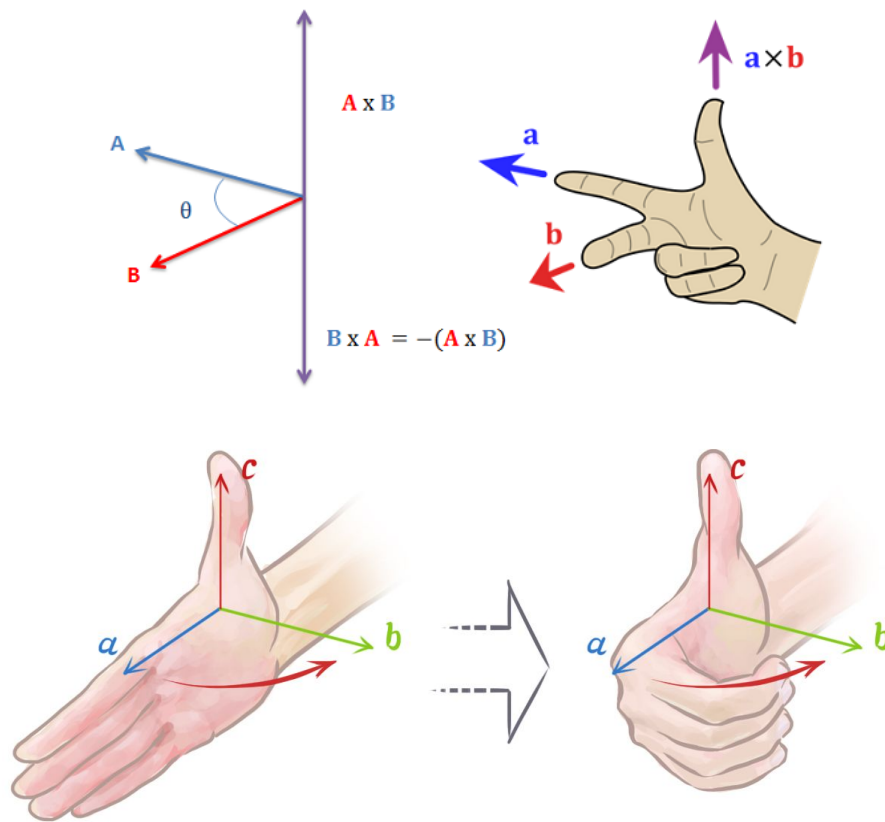
Figure 15: Ways to think about the right hand rule. In the second diagram $c = a \wedge b$.
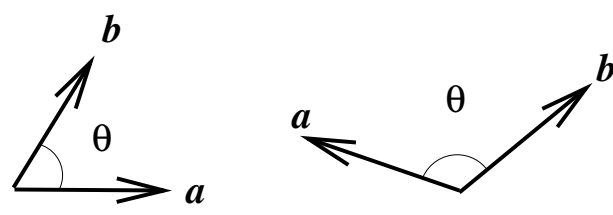


Figure 16: In the example on the left $a \wedge b$ points **out** of the page, while in the example on the right $a \wedge b$ points **into** the page.

where $\theta$ is the angle between the two vectors with $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector. The direction of $a \wedge b$, i.e. $\hat{n}$, is the direction which is perpendicular to both $a$ and $b$ in such a way that the set $a, b, a \wedge b$ forms a **right-handed**

system.

Note the following properties of $a \wedge b$:

1. The operation is *anti-commutative*,

$$a \wedge b = -b \wedge a , \qquad (38)$$

   which simply follows from the fact that the direction of the cross product reverses when the order is swapped, so as to keep a right-handed system.

2. The cross-product is distributive,

$$a \wedge (b + c) = a \wedge b + a \wedge c . \qquad (39)$$

3. The cross-product is <u>not</u> associative,

$$a \wedge (b \wedge c) \neq (a \wedge b) \wedge c , \qquad (40)$$

   we will establish this in section 1.9.

4. The cross product of two parallel vectors is zero, since then $\theta = 0$ in (37). Thus

$$\hat{i} \wedge \hat{i} = \hat{j} \wedge \hat{j} = \hat{k} \wedge \hat{k} = 0 . \qquad (41)$$

5. The vectors $\hat{i}, \hat{j}, \hat{k}$ are related via

$$\begin{aligned} \hat{i} \wedge \hat{j} &= \hat{k} , \\ \hat{j} \wedge \hat{k} &= \hat{i} , \\ \hat{k} \wedge \hat{i}. &= \hat{j} . \end{aligned} \qquad (42)$$

   These results can be checked simply using (37), noting that $\theta = \pi/2$ in each case and noting right-handedness (contrast e.g. $\hat{j} \wedge \hat{i} = -\hat{k}$).

### 1.6.1 Component formula

We can now evaluate the general vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ in components. We have

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_x \hat{\boldsymbol{i}} + a_y \hat{\boldsymbol{j}} + a_z \hat{\boldsymbol{k}}) \wedge (b_x \hat{\boldsymbol{i}} + b_y \hat{\boldsymbol{j}} + b_z \hat{\boldsymbol{k}}) \ .$$

The brackets can be expanded using the distributive property (39):

$$\begin{aligned} \boldsymbol{a} \wedge \boldsymbol{b} \ = \ & a_x b_x \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{i}} + a_x b_y \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} + a_x b_z \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{k}} \\ & + a_y b_x \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}} + a_y b_y \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{j}} + a_y b_z \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} \\ & + a_z b_x \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + a_z b_y \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{j}} + a_z b_z \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{k}} \ . \end{aligned} \tag{43}$$

We then remove all terms involving cross products of vectors with themselves using (41):

$$\boldsymbol{a} \wedge \boldsymbol{b} = a_x b_y \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} + a_x b_z \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{k}} + a_y b_x \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}} + a_y b_z \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} + a_z b_x \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + a_z b_y \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{j}} \ . \tag{44}$$

We can now employ (38) to collect terms in pairs (e.g. the one involving $\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}$ together with the one involving $\hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{i}}$):

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_y b_z - a_z b_y) \hat{\boldsymbol{j}} \wedge \hat{\boldsymbol{k}} + (a_z b_x - a_x b_z) \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} + (a_x b_y - a_y b_x) \hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} \ . \tag{45}$$

Finally, we use (42) to obtain the component form of the vector product,

$$\boxed{\boldsymbol{a} \wedge \boldsymbol{b} = (a_y b_z - a_z b_y) \hat{\boldsymbol{i}} + (a_z b_x - a_x b_z) \hat{\boldsymbol{j}} + (a_x b_y - a_y b_x) \hat{\boldsymbol{k}}} \ . \tag{46}$$

The formula (46) is quite cumbersome and hard to remember, but can be expressed much more compactly using the notation of *determinants* (see later). We write $\boldsymbol{a} \wedge \boldsymbol{b}$ as

$$\boldsymbol{a} \wedge \boldsymbol{b} \equiv \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} , \tag{47}$$

and the right-hand side is expanded using the following pattern or rules.

A second order determinant expands as:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 . \tag{48}$$

The third order determinant, which defines the vector product, then expands as:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{\boldsymbol{i}} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \hat{\boldsymbol{j}} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{\boldsymbol{k}} . \tag{49}$$

It is well worth remembering the cyclic pattern of indices in this form.

*Exercise:* Use the definition of the vector product (39) to prove the sine rule (17). [Hint: For the triangle in Figure 8, note that $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$ and take the vector product of both sides with $\boldsymbol{a}$.]

### 1.6.2 Equations for a line and plane (revisited)

We can use the vector product to derive alternative equations for a line and plane.

For instance, take the straight line we considered in Section 1.2.

As we can see, the vector $\boldsymbol{r} - \boldsymbol{a}$ is **parallel** to the vector $\boldsymbol{b} - \boldsymbol{a}$ for any point $\boldsymbol{r}$ on the line. Thus

$$(\boldsymbol{r} - \boldsymbol{a}) \wedge (\boldsymbol{b} - \boldsymbol{a}) = 0 . \tag{50}$$

This is another form for the equation of a line through the points $\boldsymbol{a}$ and $\boldsymbol{b}$.

For the equation of a plane, suppose we know three points in the plane $(A, B, C)$ as earlier.

- The vectors $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{a}$ lie in the plane.

- By the definition of the vector product, $(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a})$ is the normal (but not necessarily the unit normal) to the plane.

- For any point $r$ in the plane, the vector $r - a$ lies in the plane, and is therefore perpendicular to the normal to the plane. Using property 4 of the scalar product, it therefore follows that

$$(r - a) \cdot [(b - a) \wedge (c - a)] = 0 \tag{51}$$

which is another form for the equation of the plane passing through the points $a$, $b$ and $c$.

Example 1.9 Determine a unit vector that is perpendicular to both the vectors $(2, 1, 0)$ and $(3, 0, 1)$. Use the cross product to find the angle between these vectors

Call the first vector $a$ and the second $b$. Using the determinant formula, we have:

$$a \wedge b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix},$$
$$= \hat{i} - 2\hat{j} - 3\hat{k}.$$

This vector is, of course, perpendicular to both $a$ and $b$, but it is not a unit vector. Its magnitude is $|a \wedge b| = \sqrt{1 + 4 + 9} = \sqrt{14}$. So the perpendicular unit vector is

$$\hat{n} = \frac{1}{\sqrt{14}} \left( \hat{i} - 2\hat{j} - 3\hat{k} \right).$$

There is a second perpendicular unit vector pointing in the other direction $-\hat{n}$, which is also an acceptable answer.

Now for the angle. Let's try to use the other formula for the vector product:

$$a \wedge b = |a||b| \sin \theta \, \hat{n}.$$

We want to find $\theta$. The magnitudes are easy to compute: $|\boldsymbol{a}| = \sqrt{2^2 + 1} = \sqrt{5}$, and $|\boldsymbol{b}| = \sqrt{3^2 + 1} = \sqrt{10}$. Then we take the magnitude of both sides of the formula:

$$|\boldsymbol{a} \wedge \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta \, |\hat{\boldsymbol{n}}|,$$
$$\sqrt{14} = \sqrt{5} \cdot \sqrt{10} \cdot \sin \theta \cdot 1,$$

noting that $\sin \theta$ is positive because $\theta$ is between $0$ and $\pi$. And we solve for $\theta$:

$$\theta = \sin^{-1}\left(\frac{\sqrt{7}}{5}\right) \approx 32°, \, 148°.$$

Note that the inverse sin is multivalued and yields two angles between $0$ and $\pi$! This is a problem when trying to determine the angle between two vectors using the vector product. It is much safer to use the scalar product to get this angle, and also it is usually less work as well! Let's now do that. We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta,$$
$$(2, 1, 0) \cdot (3, 0, 1) = \sqrt{5}\sqrt{10} \cos \theta,$$
$$\rightarrow \cos \theta = 6/\sqrt{50},$$

and thus $\theta = \cos^{-1}(6/\sqrt{50}) \approx 32°$, on the range between $0$ and $\pi$. So the smaller of the two angles is the correct one.

### Example 1.10

A rigid body spins with angular speed $\omega$ about an axis $\hat{\boldsymbol{n}}$. Find a vector expression for the velocity of any point $P$ in the body.

(The speed of the motion of any point is equal to the angular speed $\omega$ multiplied by the distance $\rho$ between the point and the axis about which it is rotating.)

Let the velocity of point $P$ be $\boldsymbol{u}$. Let us determine its magnitude first $|\boldsymbol{u}|$. From trigonometry we have

$$|\boldsymbol{u}| = \rho\omega = (|\boldsymbol{p}| \sin \theta)\omega = |\hat{\boldsymbol{n}} \wedge \boldsymbol{p}|\omega.$$
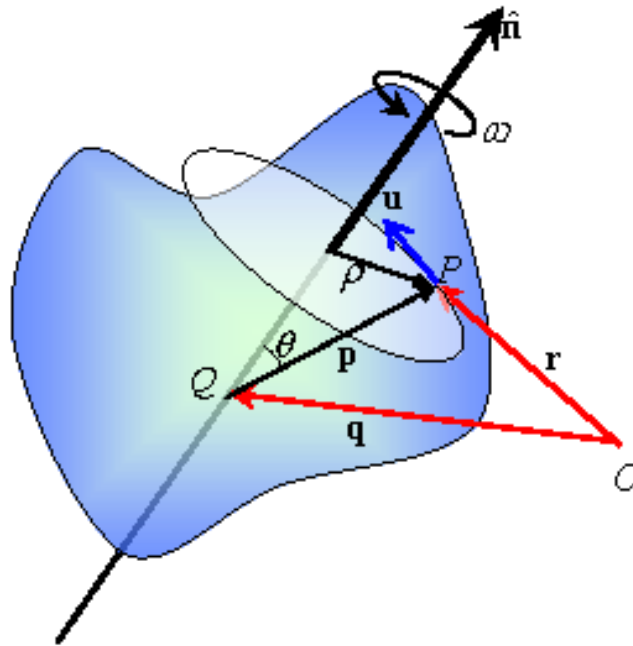
Figure 17: An irregularly shaped body spinning about the axis $\hat{\boldsymbol{n}}$ at a rate $\omega$. Here $P$ is a random point within the body and $O$ is the origin.

What about the direction of the motion, $\hat{\boldsymbol{u}}$? This direction is perpendicular to both the vector $\boldsymbol{p}$ and the rotation axis $\hat{\boldsymbol{n}}$, thus is parallel to $\hat{\boldsymbol{n}} \wedge \boldsymbol{p}$. The unit vector in this direction is hence $\hat{\boldsymbol{u}} = (\hat{\boldsymbol{n}} \wedge \boldsymbol{p})/|\hat{\boldsymbol{n}} \wedge \boldsymbol{p}|$. Putting everything together

$$\boldsymbol{u} = |\boldsymbol{u}|\hat{\boldsymbol{u}} = (\hat{\boldsymbol{n}} \wedge \boldsymbol{p})\omega,$$
$$= [\hat{\boldsymbol{n}} \wedge (\boldsymbol{r} - \boldsymbol{q})]\omega.$$

## 1.7 Calculating distances with vectors

### 1.7.1 Shortest distance of a point from a line

Consider the line given by $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$, and the point $Q$ with position vector $\boldsymbol{q}$.
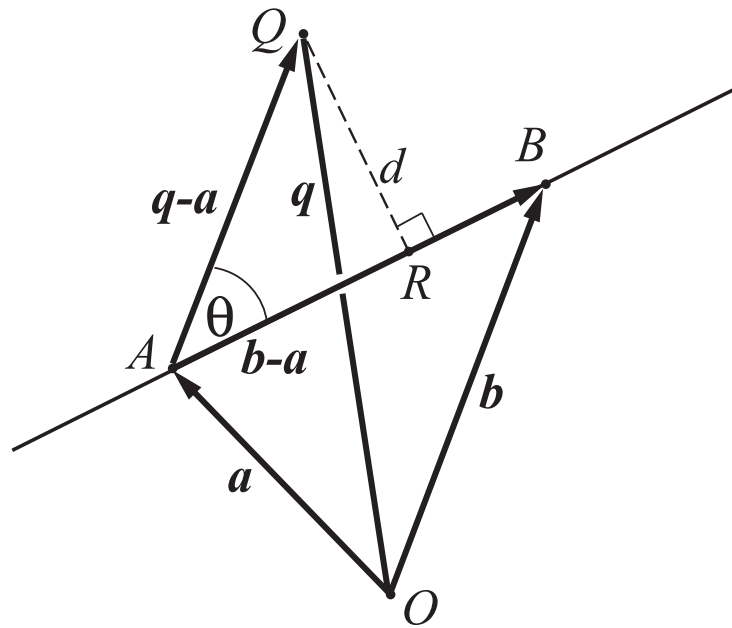


Figure 18: Shortest distance of the point $Q$ from the line $AB$.

The shortest distance, $d$, between $Q$ and the line is perpendicular to the line, i.e $QR$ in the figure. Considering the right-angled triangle $AQR$, we see that

$$d = |\boldsymbol{q} - \boldsymbol{a}| \sin \theta \ ,$$

with $\theta$ the angle between $\overrightarrow{AQ} = \boldsymbol{q} - \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$. Using the definition (37) we can therefore write

$$\boxed{d = \frac{|(\boldsymbol{q} - \boldsymbol{a}) \wedge (\boldsymbol{b} - \boldsymbol{a})|}{|\boldsymbol{b} - \boldsymbol{a}|}} \ . \tag{52}$$

If we use as our equation of the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\hat{\boldsymbol{t}}$, where $\hat{\boldsymbol{t}}$ is a unit vector, then (52) simplifies to

$$\boxed{d = |(\boldsymbol{q} - \boldsymbol{a}) \wedge \hat{\boldsymbol{t}}\,|} . \tag{53}$$

### 1.7.2 Shortest distance of a point from a plane

Consider the plane with equation $(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0$, and consider the point $P$ with position vector $\boldsymbol{p}$.
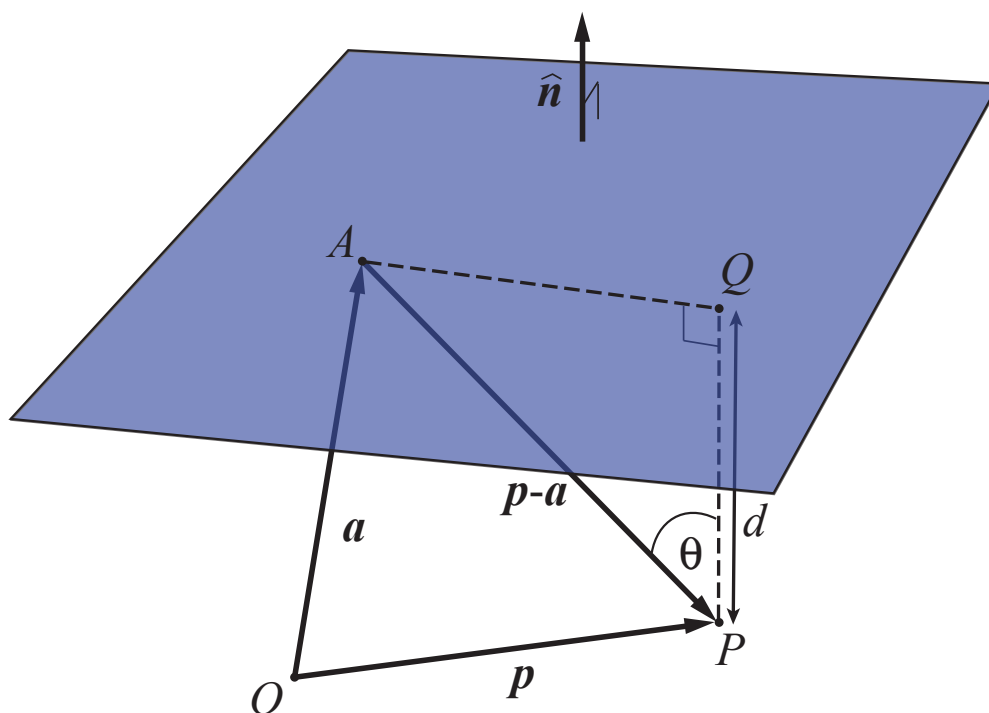


Figure 19: Shortest distance of the point $P$ from a plane.

- We draw the line (not necessarily in the plane) from $P$ that is perpendicular to the plane. This line meets the plane at $Q$. The shortest distance from the point $P$ to the plane is the distance $PQ$.
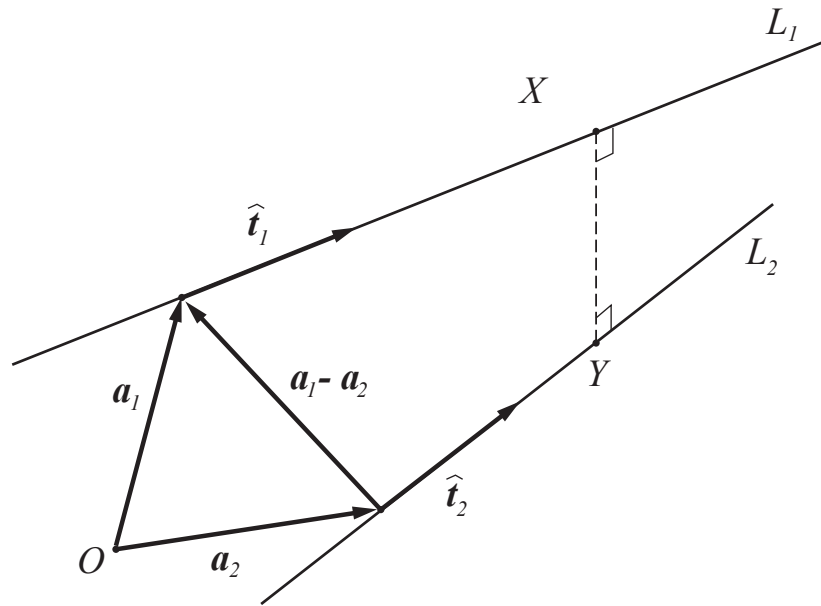
Figure 20: Non-intersecting lines $L_1$ and $L_2$ in three dimensions.

- We consider the right-angled triangle $APQ$. Then

$$d \equiv |\overrightarrow{PQ}| = |\overrightarrow{AP}| \cos\theta \,, \tag{54}$$

and recalling the definition of the scalar product (23) we find that the shortest distance is given by

$$\boxed{d = |(\boldsymbol{p} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}|} \,. \tag{55}$$

### 1.7.3 Shortest distance of a line from a line

Finally, we consider the shortest distance between two (non-parallel) lines $L_1$ and $L_2$, given by

$$\boldsymbol{r} = \boldsymbol{a_1} + \lambda\hat{\boldsymbol{t_1}}, \qquad \text{and} \qquad \boldsymbol{r} = \boldsymbol{a_2} + \mu\hat{\boldsymbol{t_2}},$$

respectively, for $\lambda$ and $\mu$ scalars.

- Suppose that the shortest distance between the two lines is between the points $X$ and $Y$. Since this is the shortest distance, the line joining $X$ and $Y$ will be perpendicular to both lines.

- The first question is: What direction is $\overrightarrow{XY}$ in? The two lines are parallel to $\hat{t}_1$ and $\hat{t}_2$ respectively, and $\overrightarrow{XY}$ is perpendicular to both of these. The vector which is perpendicular to both lines is simply $u = \hat{t}_1 \wedge \hat{t}_2$. We may then write $\overrightarrow{XY} = d\hat{u}$, in which we have normalised $u$ to get a unit vector. Our goal is to find $d$.

- Second thing: let us rewrite $\overrightarrow{XY}$ using the triangle $XYO$:

$$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = (a_2 - a_1) + \mu\hat{t}_2 - \lambda\hat{t}_1,$$

for some specific scalars $\lambda$ and $\mu$ (which we don't know yet).

- We have two expressions now for $\overrightarrow{XY}$. We equate them, then $\cdot\hat{u}$ both sides of the equation. We obtain simply

$$\boxed{d = \frac{|(a_2 - a_1) \cdot (\hat{t}_1 \wedge \hat{t}_2)|}{|\hat{t}_1 \wedge \hat{t}_2|}}. \tag{56}$$

Equation (56) gives us an easy way of finding out if two lines intersect or not. If they do intersect then their closest distance is obviously zero, i.e. $d = 0$. The condition for two lines to intersect is therefore

$$(a_2 - a_1) \cdot \hat{t}_1 \wedge \hat{t}_2 = 0 \,. \tag{57}$$

## 1.8 Scalar triple product

The quantity

$$a \cdot (b \wedge c) \tag{58}$$

is a scalar, and is called the *scalar triple product* of $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$. The *only* way that this product makes sense is if

- the vector product is taken first, to give the vector $\boldsymbol{b} \wedge \boldsymbol{c}$,

- and then the scalar product is taken between $\boldsymbol{a}$ and $\boldsymbol{b} \wedge \boldsymbol{c}$.

For this reason the brackets are unnecessary, and often omitted, so we write $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$. Note also that the notation $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is sometimes used.

### 1.8.1 Component formula

In terms of components (instead of angles and magnitudes), the scalar triple product equals

$$\boxed{\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x)} , \tag{59}$$

where $\boldsymbol{a} = (a_x, a_y, a_z)$, etc.

The formula (59) is hard to remember, but can be expressed much more compactly using the notation of determinants introduced already. We write

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} \equiv \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} , \tag{60}$$

which has been formed from taking the components of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as the three rows. The determinant is now expanded using the pattern shown below (see also eq. (49)):

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} . \tag{61}$$

By rearranging this formula, we can find the following results:

$$\boxed{\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} &= \boldsymbol{b} \cdot \boldsymbol{c} \wedge \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{a} \wedge \boldsymbol{b} \\ &= -\boldsymbol{a} \cdot \boldsymbol{c} \wedge \boldsymbol{b} = -\boldsymbol{b} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = -\boldsymbol{c} \cdot \boldsymbol{b} \wedge \boldsymbol{a} \end{aligned}} . \tag{62}$$

In other words the value of $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$ is unchanged if we make an *even* permutation of the order of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, but the sign changes if we make an *odd* permutation of the order of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

**Permutations (aside)**

- Suppose we have an ordered sequence of elements (such as $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$).

- A permutation is a rearrangement of the sequence in which the positions of some of the elements are swapped.

- An *even* permutation involves an *even* number of swaps of two elements, e.g. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rightarrow \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}$ is even because we swap $\boldsymbol{a}$ and $\boldsymbol{c}$, and then swap $\boldsymbol{b}$ and $\boldsymbol{c}$ (two swaps). An *odd* permutation is when we make an *odd* number of swaps.

- For a sequence of three elements, an even permutation is equivalent to a *cyclic* permutation, e.g. $\boldsymbol{a} \rightarrow \boldsymbol{b}$, $\boldsymbol{b} \rightarrow \boldsymbol{c}$, $\boldsymbol{c} \rightarrow \boldsymbol{a}$.

Example 1.11 Evaluate the scalar triple product of the vectors $(2, 3, 1)$, $(1, 2, 2)$ and $(4, 5, 1)$.

We write the triple product in the form of a determinant:

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 4 & 5 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix},$$

$$= 2(2 \cdot 1 - 2 \cdot 5) + 3(2 \cdot 4 - 1 \cdot 1) + (1 \cdot 5 - 2 \cdot 4),$$

$$= 2.$$
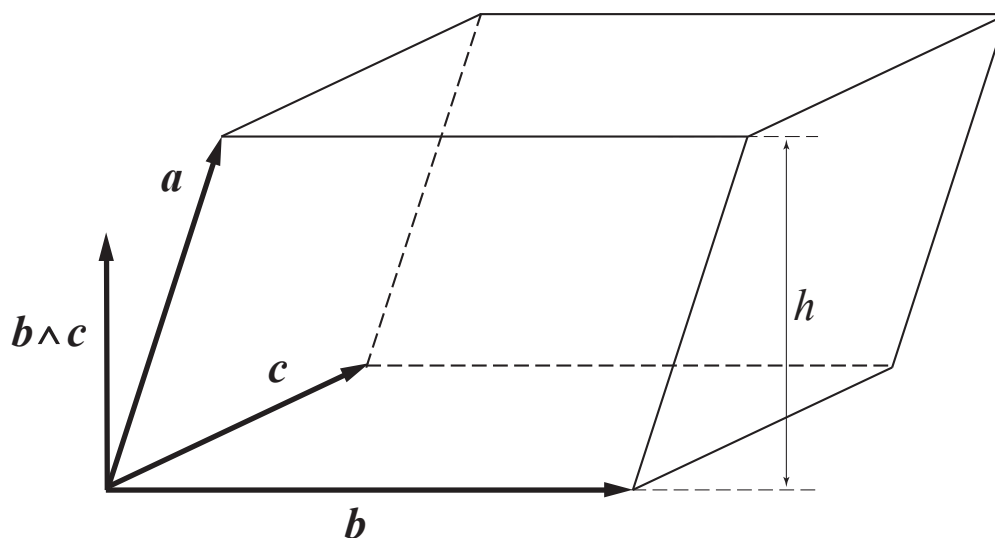
Figure 21: Parallelepiped formed from the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

### 1.8.2 Parallelepipeds

One important application of the scalar triple product is the calculation of the volume of a *parallelepiped.*

Consider the parallelepiped formed by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (see Figure 21). The area of the base is

$$|\boldsymbol{b}||\boldsymbol{c}| \sin \theta = |\boldsymbol{b} \wedge \boldsymbol{c}| . \tag{63}$$

The height $h$ of the parallelepiped corresponds to the component of the vector $\boldsymbol{a}$ in the direction perpendicular to the base.

The unit vector perpendicular to the base is

$$\frac{\boldsymbol{b} \wedge \boldsymbol{c}}{|\boldsymbol{b} \wedge \boldsymbol{c}|}, \tag{64}$$

and therefore the height $h$ is

$$\boldsymbol{a} \cdot \frac{\boldsymbol{b} \wedge \boldsymbol{c}}{|\boldsymbol{b} \wedge \boldsymbol{c}|} . \tag{65}$$

The volume of the parallelepiped is the area of its base times its height, and putting together (63) and (65) we find that the volume is

$$\text{Volume} = |\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}| , \tag{66}$$

i.e. the scalar triple product.

Example 1.12 Show that $\boldsymbol{a} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = \boldsymbol{0}$. What does this result mean geometrically? What is the condition for three vectors to be coplanar?

We just permute the elements in the triple product:

$$\boldsymbol{a} \cdot \boldsymbol{a} \wedge \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a} \wedge \boldsymbol{a} = 0$$

the last equality coming from the fact that a vector product of a vector with itself is zero. Of course, this is all obvious because a vector product involving $\boldsymbol{a}$ will always give you a vector perpendicular to $\boldsymbol{a}$.

A geometric interpretation of this is that the parallelepiped formed from $\boldsymbol{a}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ has zero volume because it is necessarily flat, i.e. a 2D object.

We can then use this idea to derive a test to see if any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are coplanar. If the three vectors are coplanar (i.e. lying in the same plane) then the parallelepiped they form is flat and has zero area, or in mathematical terms:

$$\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{0}.$$

Example 1.13 Find the volume of the rhomboid drawn in the face-centred cubic in Figure 22.

A rhomboid is a special case of a parallelepiped. Let us work out the three vectors that define it. First note that $\overrightarrow{OA} = (a, 0, 0)$, $\overrightarrow{OB} = (a/2, a/2, 0)$,
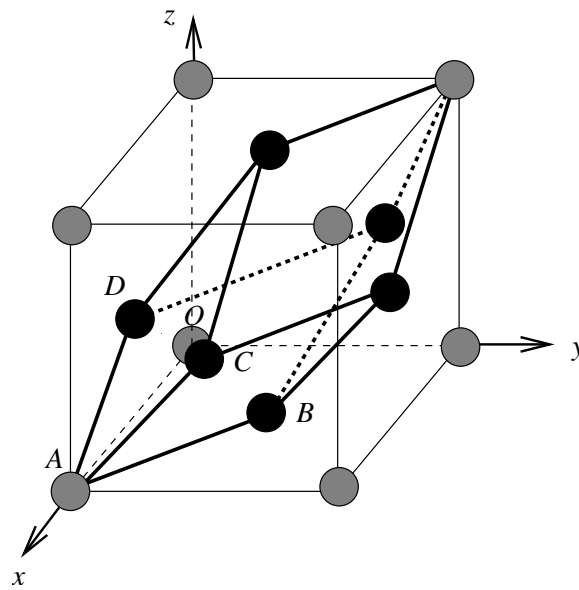
**Figure 22:** Cubic lattice, side length $a$. The black atoms are located at the centres of each face and the grey atoms at the corners.

$\overrightarrow{OC} = (a, a/2, a/2)$, and $\overrightarrow{OD} = (a/2, 0, a/2)$. The three defining vectors of the rhomboid are:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-a/2, a/2, 0),$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (0, a/2, a/2),$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-a/2, 0, a/2).$$

The volume is just the scalar triple product of these vectors:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \wedge \overrightarrow{AD} = \begin{vmatrix} -a/2 & a/2 & 0 \\ 0 & a/2 & a/2 \\ -a/2 & 0 & a/2 \end{vmatrix},$$

$$= -(a/2) \begin{vmatrix} a/2 & a/2 \\ 0 & a/2 \end{vmatrix} - (a/2) \begin{vmatrix} 0 & a/2 \\ -a/2 & a/2 \end{vmatrix} + 0,$$

$$= -(a/2)(a/2)^2 - (a/2)(a/2)^4 = -a^3/4.$$

The volume is the absolute value of this (the minus sign comes about because the three vectors defining the rhomboid are left handed).

## 1.9 Vector triple product

The quantity $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ is a vector, and is called the *vector triple product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Unlike $\boldsymbol{a} \cdot \boldsymbol{b} \wedge \boldsymbol{c}$, the position of the brackets in $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ is crucial! We can use our expression for the vector product in components to expand $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ in components. First, use (46) to evaluate $\boldsymbol{b} \wedge \boldsymbol{c}$, so that

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}}) \wedge \Big[ (b_yc_z - b_zc_y)\hat{\boldsymbol{i}} \\ + (b_zc_x - b_xc_z)\hat{\boldsymbol{j}} + (b_xc_y - b_yc_x)\hat{\boldsymbol{k}} \Big]$$

and then using the distributive property (39) to expand the brackets and the results (41) and (42) to evaluate the cross products, we find

$$\begin{aligned} \boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) &= (-a_y\hat{\boldsymbol{k}} + a_z\hat{\boldsymbol{j}})(b_yc_z - b_zc_y) + (a_x\hat{\boldsymbol{k}} - a_z\hat{\boldsymbol{i}})(b_zc_x - b_xc_z) \\ &+ (-a_x\hat{\boldsymbol{j}} + a_y\hat{\boldsymbol{i}})(b_xc_y - b_yc_x) \, . \end{aligned}$$

gathering the terms in $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ we obtain the final result

$$\boxed{\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}} \, . \tag{67}$$

Essentially, this is a kind of projection of $\boldsymbol{a}$ down onto the plane formed be the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

We can proceed in exactly the same manner to find $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ in the form

$$\boxed{(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}} \, . \tag{68}$$

By comparing (67) and (68) it becomes clear that $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) \neq (\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ in general.

<u>Example 1.14</u> Calculate $(\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}) \wedge \hat{\boldsymbol{i}}$ in two ways. Calculate $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{c} \wedge \boldsymbol{d})$.

<span style="color:red">One way uses the identities $\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$. Simply:</span>

$$\color{red}(\hat{\boldsymbol{i}} \wedge \hat{\boldsymbol{j}}) \wedge \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \, .$$

A different way uses the general identity for triple vector products:

$$(\hat{i} \wedge \hat{j}) \wedge \hat{i} = (\hat{i} \cdot \hat{i})\hat{j} - (\hat{i} \cdot \hat{j})\hat{i} = \hat{j}$$

A third way computes the vector products using the determinant formula and coordinates (noting that $\hat{j} = (0, 1, 0)$, etc.) This is a boring way, which we will leave to one side.

The last part of the question deals with a daunting vector quadruple product. This can be transformed into a triple product by setting $r = c \wedge d$, then

$$
\begin{aligned}
(a \wedge b) \wedge (c \wedge d) &= (a \wedge b) \wedge r, \\
&= (a \cdot r)b - (r \cdot b)a, \\
&= [a \cdot (c \wedge d)]b - [b \cdot (c \wedge d)]a,
\end{aligned}
$$

an expression that involves two scalar triple products.

But equally we could have set $s = a \times b$ and proceeded to derive a similar but different expression:

$$
\begin{aligned}
(a \wedge b) \wedge (c \wedge d) &= s \wedge (c \wedge d), \\
&= (s \cdot d)c - (s \cdot c)d, \\
&= [d \cdot (a \wedge b)]c - [c \cdot (a \wedge b)]d.
\end{aligned}
$$

We can equate these two expressions and get something like:

$$\alpha a + \beta b = \gamma c + \delta d$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are scalar triple products. This is basically saying that, in 3D, there must be a linear relation between any four vectors. In other words, given four vectors in 3D, you can generally write one of those vectors in terms of the other three.

## 1.10 Vector area

We are familiar with the concept of the area of a surface as being a scalar. However, in some applications, particularly with integrals of vector quantities such as electric or magnetic flux through a surface, we will also want to think about the *vector area.*

Consider a flat planar surface of area $A$, with unit normal $\hat{\boldsymbol{n}}$, then the vector area $\boldsymbol{S}$ is defined by

$$\boxed{\boldsymbol{S} = A\hat{\boldsymbol{n}}} \,. \tag{69}$$

In terms of Cartesian components, we can write

$$
\begin{aligned}
\boldsymbol{S} &\equiv S_x\hat{\boldsymbol{i}} + S_y\hat{\boldsymbol{j}} + S_z\hat{\boldsymbol{k}} \\
&= A\Big[(\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{i}} + (\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{j}} + (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{k}}\Big]
\end{aligned}
\tag{70}
$$

where the second step has been taken by simply taking the scalar product of $\boldsymbol{S}$ with each axis in turn. The quantities $S_x, S_y, S_z$ are just the areas of the projections of the patch onto the planes $x = 0, y = 0, z = 0$ respectively.
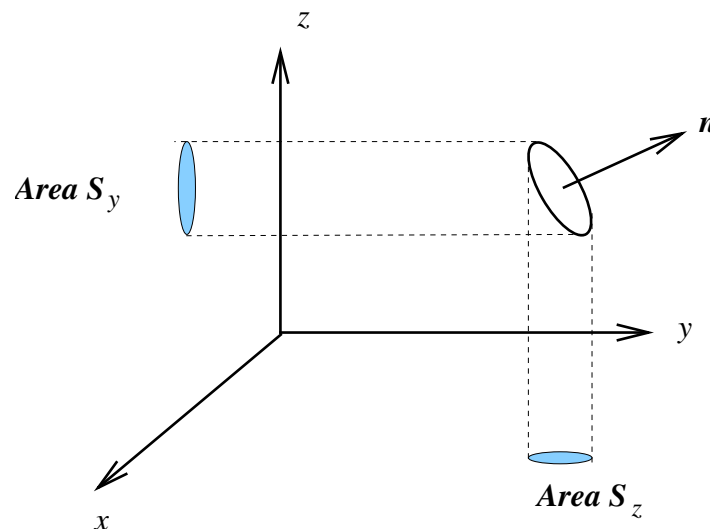


Figure 23: Vector area $\boldsymbol{S}$ and its projections on to the $xz$ and $xy$ planes.

Given any old surface, there is an ambiguity in which direction the normal should point. However, for a closed surface, such as a cube, one always takes the enclosing surfaces to have normals pointing *out* of the volume enclosed.

The vector area of the whole surface is then taken to be the sum over the vector areas of each individual patch. **For a closed surface it turns out the vector area is zero.** (We will not prove this result here.)
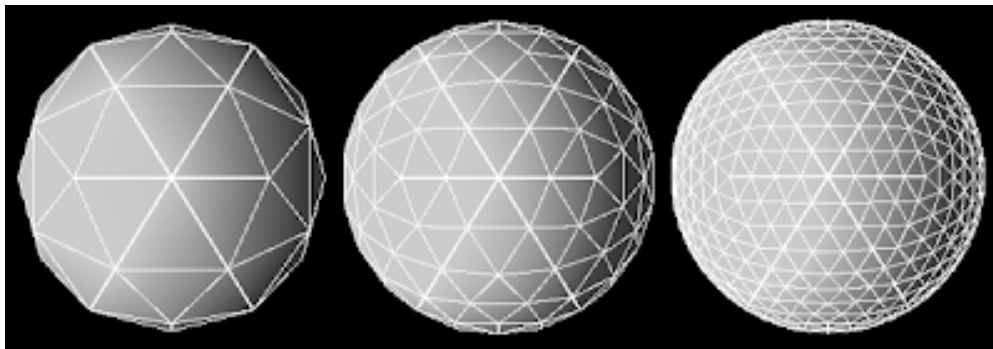


Figure 24: Successive polyhedral approximations to the surface of a sphere

We can also approximate the vector area of a *curved* surface (such as a sphere) by splitting the surface into small plane patches. In the limit of these patches shrinking to zero size the finite sum is then replaced by an integral over the whole surface to give the exact vector area. Calculating such surface integrals will be described in the course next term, but certain examples are possible now using the fact that the vector area of a closed surface vanishes.

Example 1.16

Calculate the vector area of the pyramid with vertices $O = (0,0,0)$, $A = (0,1,0)$, $B = (1,1,0)$, $C = (1,0,0)$ and $D = (\frac{1}{2}, \frac{1}{2}, 1)$, excluding the base.

Let us first work out the vector area of the triangle $OAD$. To get the normal
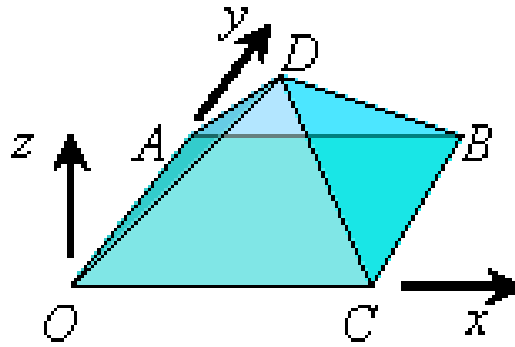
to this surface, we calculate $\boldsymbol{n} = \overrightarrow{OD} \wedge \overrightarrow{OA}$:

$$\boldsymbol{n} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1/2 & 1/2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\hat{\boldsymbol{i}} + \tfrac{1}{2}\hat{\boldsymbol{k}}. \tag{71}$$

But we want the unit normal. The magnitude of $\boldsymbol{n}$ is $\sqrt{1 + 1/4} = \sqrt{5}/2$, and so the unit normal to this surface is:

$$\hat{\boldsymbol{n}} = \frac{2}{\sqrt{5}}(-1, 0, \tfrac{1}{2}).$$

What is the scalar area of $OAD$? This is actually

$$(1/2)|\overrightarrow{OD} \wedge \overrightarrow{OA}| = (1/2)|\boldsymbol{n}| = \frac{\sqrt{5}}{4}.$$

So finally the vector area of $OAD$ is

$$\boldsymbol{S}_{\text{OAD}} = \frac{\sqrt{5}}{4}\hat{\boldsymbol{n}} = \left(-\frac{1}{2}, 0, \frac{1}{4}\right).$$

Doing similar calculations we can find the vector area of the other triangular faces: $OCD$, $CBD$, and $ABD$:

$$\boldsymbol{S}_{\text{OCD}} = (0, -\tfrac{1}{2}, \tfrac{1}{4}), \quad \boldsymbol{S}_{\text{CBD}} = (\tfrac{1}{2}, 0, \tfrac{1}{4}), \quad \boldsymbol{S}_{\text{ABD}} = (0, \tfrac{1}{2}, \tfrac{1}{4}).$$

The total vector area of the pyramid (without base) is the sum of these four triangular vector areas. We find that $\mathbf{S} = (0, 0, 1) = \hat{\boldsymbol{k}}$.

Alternatively: if we do include the base, then the total vector area of the closed pyramidal surface is zero, i.e.

$$\boldsymbol{S}_{\text{pyramid}} + \boldsymbol{S}_{\text{base}} = \boldsymbol{0}.$$

The vector area of the base is easy: its scalar area is 1 and it points vertically downwards. Thus $\boldsymbol{S}_{\text{base}} = -\hat{\boldsymbol{k}}$, and so $\boldsymbol{S}_{\text{pyramid}} = \hat{\boldsymbol{k}}$.

Example 1.17 Calculate the vector area of a hemispherical shell of radius $a$, oriented so that its circular base lies in the $xy$ plane and the shell itself is above the plane.

The total vector area of the hemisphere and its circular base is *zero*, because such a figure is a closed volume:

$$\boldsymbol{S}_{\text{total}} = \boldsymbol{S}_{\text{hemi}} + \boldsymbol{S}_{\text{base}} = \boldsymbol{0}.$$

The vector area of the hemisphere is difficult to calculate directly, but the base is easy as it is a circle of radius $a$ and its normal pointing out of the volume is $-\hat{\boldsymbol{k}}$. Therefore: $\boldsymbol{S}_{\text{base}} = -\pi a^2 \hat{\boldsymbol{k}}$, which means that $\boldsymbol{S}_{\text{hemi}} = \pi a^2 \hat{\boldsymbol{k}}$.

## 1.11  General basis vectors

In three-dimensional space any three *non-coplanar* vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ constitute a *basis*

This means that any vector can be expressed as a linear combination of these three basis vectors

$$\boldsymbol{u} = \alpha \boldsymbol{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c},$$

where $\boldsymbol{u}$ is any vector and $\alpha$, $\beta$, and $\gamma$ are unique scalars: they are called $\boldsymbol{u}$'s components with respect to this specific basis.

It is not necessary for the basis vectors to be perpendicular to each other. However, if they are, the basis is said to be **orthogonal**, and if the basis vectors have unit length as well then the basis is **orthonormal**. The Cartesian basis $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ is orthonormal.

In the special situation of an orthogonal basis, $\alpha$, $\beta$, and $\gamma$ can be obtained by the scalar product. We find $\alpha = (\boldsymbol{u}\cdot\boldsymbol{a})/|\boldsymbol{a}|^2$, $\beta = (\boldsymbol{u}\cdot\boldsymbol{b})/|\boldsymbol{b}|^2$, $\gamma = (\boldsymbol{u}\cdot\boldsymbol{c})/|\boldsymbol{c}|^2$. However, for non-orthogonal basis sets this is not true and the components $\alpha$, $\beta$, and $\gamma$ can be obtained in terms of scalar triple products (see worked example later).

## 1.12   Other orthogonal coordinate systems

The Cartesian $xyz$ coordinates are a very familiar example of an orthogonal coordinate system in three dimensions. However, there are other examples which can be very useful in particular applications, and we will describe the two most commonly used ones here.

### 1.12.1   Cylindrical polar coordinates

We introduce the cylindrical polar coordinates $(r, \theta, z)$.

- The $z$ coordinate is the same as for Cartesian coordinates,

- the $r$ coordinate corresponds to the distance $OQ$

- the azimuthal angle $\theta$ corresponds to the angle between the $x$ axis and the vector $\overrightarrow{OQ}$. By convention we take $0 \leqslant \theta < 2\pi$, but an alternative is $-\pi < \theta \leqslant \pi$.
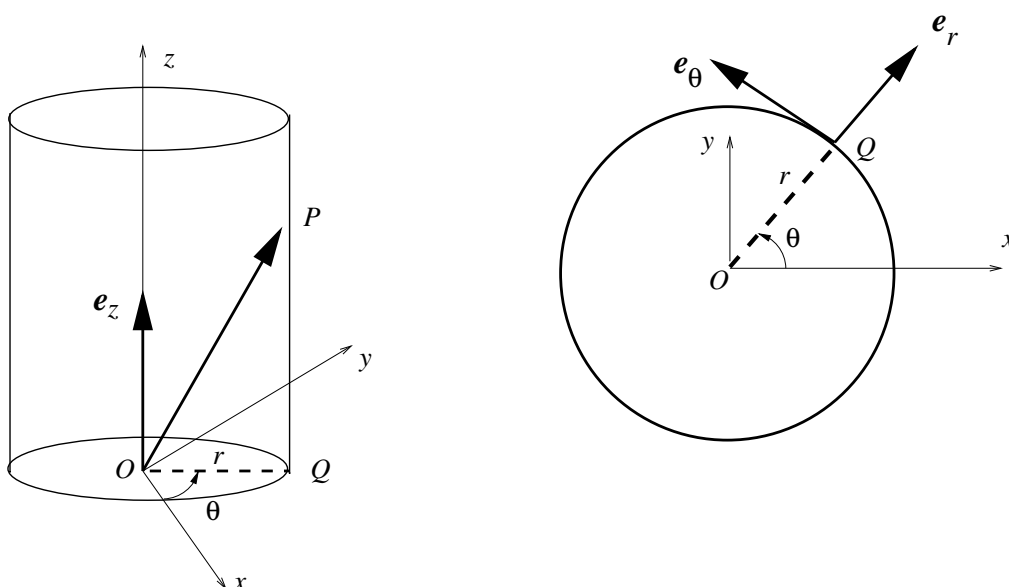
Figure 25: Diagrams illustrating the cylindrical polar coordinate system.

- The general point $P$ has position vector $\boldsymbol{r} = (x, y, z)$ in Cartesians. Via simple trigonometry its $x$ and $y$ components are

$$x = r\cos\theta \qquad y = r\sin\theta \,, \tag{72}$$

so that

$$\boxed{\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{i}} + r\sin\theta\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}} \,. \tag{73}$$

- As well as new coordinates $(r, \theta, z)$ we introduce new coordinate directions, or rather unit vectors $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_z$, defined via

$$\boxed{\begin{aligned} \hat{\boldsymbol{e}}_r &= \cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}} \\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}} \\ \hat{\boldsymbol{e}}_z &= \hat{\boldsymbol{k}} \end{aligned}} \tag{74}$$

The three unit vectors $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_z$ are perpendicular to each other (check that the scalar products are zero) and form a right-handed orthonormal system. In particular (check using 74)

$$\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_r \wedge \hat{\boldsymbol{e}}_\theta \quad \hat{\boldsymbol{e}}_r = \hat{\boldsymbol{e}}_\theta \wedge \hat{\boldsymbol{e}}_z \quad \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{e}}_z \wedge \hat{\boldsymbol{e}}_r \,.$$

Watch out, these vectors are not constant vectors! They depend on position!

Finally,

$$|\boldsymbol{r}| = \sqrt{r^2 + z^2} \, , \tag{75}$$

and in terms of the new cylindrical basis vectors (rather than the Cartesian basis vectors):

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r + z\hat{\boldsymbol{e}}_z \, . \tag{76}$$

### 1.12.2 Plane polar coordinates

If we just work in the $xy$ plane, i.e. in two dimensions, then we can use plane polar coordinates $(r, \theta)$. These are simply cylindrical polar coordinates but with the $z$ direction suppressed. So for instance in two dimensions

$$\boxed{\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{i}} + r\sin\theta\hat{\boldsymbol{j}}} \, . \tag{77}$$

We have seen this 2D parametrisation already in eq. (11).

### 1.12.3 Spherical coordinates

We next introduce the spherical polar coordinates $(r, \theta, \phi)$.

- The $r$ coordinate is the distance of the point $P$ from the origin.

- The co-latitude $\theta$ coordinate is the angle between the upward vertical axis and the line $\overrightarrow{OP}$, measured in the direction *down* from the North Pole. So that the North Pole has $\theta = 0$, the equator has $\theta = \pi/2$, and the South Pole has $\theta = \pi$.

- The azimuthal angle $\phi$ coordinate is the angle measured from the fixed $x$ axis round to the direction $\overrightarrow{QP}$ in the anticlockwise sense.

- It is important to note the ranges of the angular coordinates:

$$\boxed{\begin{array}{ccc} 0 & \leqslant \theta & \leqslant \pi \\ 0 & \leqslant \phi & < 2\pi \end{array}} .$$

- Note that $r$, $\boldsymbol{\theta}$ in spherical polars are different from $r$, $\boldsymbol{\theta}$ in cylindrical polars (with the latter often denoted $\boldsymbol{\rho}$, $\boldsymbol{\phi}$ to avoid confusion).

- The general point $P$ has position vector $\boldsymbol{r}$; from the right-angled triangle $OPQ$ we see that for $P$

$$|PQ| = r\sin\theta \quad , \qquad |OQ| = z = r\cos\theta . \tag{78}$$

Then, via simple trigonometry in the $xy$ plane

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \end{aligned} , \tag{79}$$

so that

$$\boxed{\boldsymbol{r} = r\sin\theta\cos\phi\hat{\boldsymbol{i}} + r\sin\theta\sin\phi\hat{\boldsymbol{j}} + r\cos\theta\hat{\boldsymbol{k}}} . \tag{80}$$
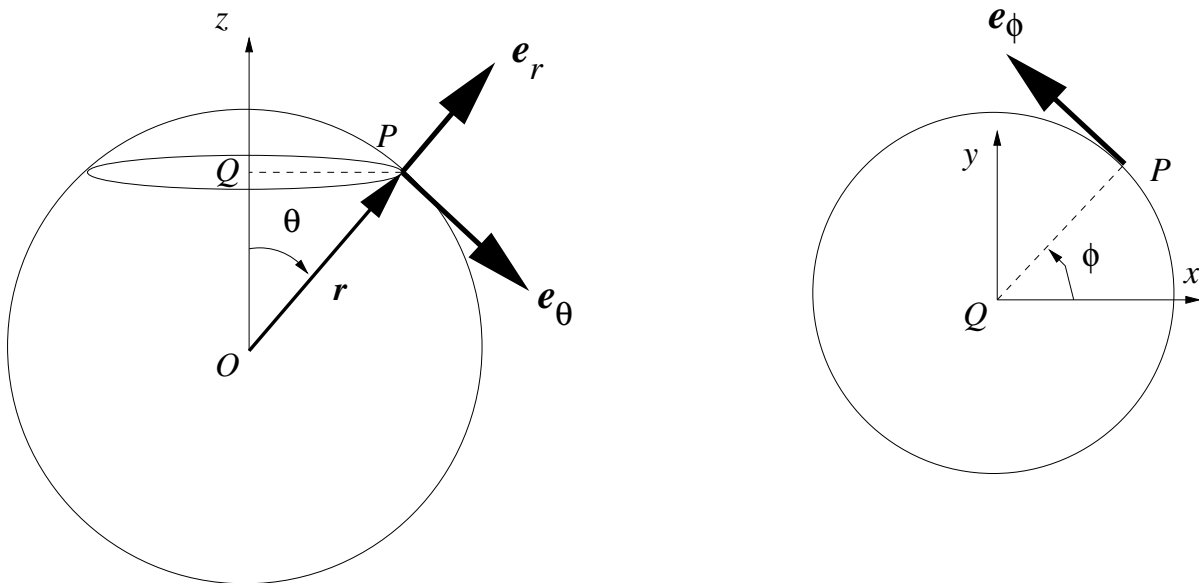


Figure 26: Diagrams illustrating the spherical coordinate system.

- We can introduce associated unit vectors: $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. These point in the direction of increasing $r$, $\theta$, and $\phi$, respectively.

  The spherical polar unit vectors can be written in terms of $\hat{i}, \hat{j}, \hat{k}$. To do this, first note that the vector $\overrightarrow{QP}$ is parallel to $\cos\phi\hat{i} + \sin\phi\hat{j}$. Next we re-express $\hat{e}_r$ as the sum of its projection in the direction $\overrightarrow{QP}$ and its projection in $\hat{k}$. We do the same with $\hat{e}_\theta$. We find:

  $$\boxed{\begin{aligned} \hat{e}_r &= \sin\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) + \cos\theta\hat{k} \\ \hat{e}_\theta &= \cos\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) - \sin\theta\hat{k} \end{aligned}} \tag{81}$$

  Finally, in the $xy$ plane we resolve $\hat{e}_\phi$ in the directions of $\hat{i}$ and $\hat{j}$ to find

  $$\boxed{\hat{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j}} \tag{82}$$

- The three unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are perpendicular to each other, and form a right-handed orthonormal system. In particular,

  $$\hat{e}_\phi = \hat{e}_r \wedge \hat{e}_\theta \quad \hat{e}_r = \hat{e}_\theta \wedge \hat{e}_\phi \quad \hat{e}_\theta = \hat{e}_\phi \wedge \hat{e}_r .$$

  Finally, note that in spherical polar basis vectors the position vector of $P$ is very simple:

  $$\boldsymbol{r} = r\hat{e}_r . \tag{83}$$

Example 1.18 Calculate the coordinates of the point $(1, 1, 2)$ in (a) cylindrical polars, (b) spherical polars.

(a) In cylindrical polars, radius is given by $r^2 = x^2 + y^2 = 2$ and so $r = \sqrt{2}$, whereas azimuthal angle is given by $\tan\theta = y/x = 1$, which means $\theta = \pi/4$. The vertical height $z$ is unchanged.

(b) In spherical polars, radius is given by $r^2 = x^2 + y^2 + z^2 = 1 + 1 + 2^2$ and so $r = \sqrt{6}$.

We also have $z = r\cos\theta$ which lets us get the colatitude $\theta$: $\cos\theta = 2/\sqrt{6}$, which means $\theta = \cos^{-1}\sqrt{2/3} \approx 35°$.

Finally, $x = r\sin\theta\cos\phi$, which yields the azimuthal angle $\phi$:

$$\cos\phi = \frac{x}{r\sin\theta} = \frac{x}{r\sqrt{1 - \cos^2\theta}} = \frac{1}{\sqrt{6}\sqrt{1 - 2/3}} = \frac{1}{\sqrt{2}},$$

and so $\phi = \cos^{-1}\sqrt{1/2} = \pi/4 = 45°$.

Example 1.19 [Tripos question 2005 Paper 1, 5C]

The vertices of a tetrahedron $O, P, Q, R$ have coordinates $(0,0,0)$, $(2,1,1)$, $(1,2,2)$ and $(0,0,3)$ respectively. Find by vector methods:

(a) the angle between the faces $OPR$ and $OQR$

(b) the angle between the vector $OP$ and the normal to the face $PQR$

(c) the area of the face $PQR$

(d) the shortest distance from the origin to the plane containing $P, Q$ and $R$.

(a) The triangular surfaces $OPR$ and $OQR$ define two planes with unit normals $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, respectively. The angle between two planes is equal to the angle between their normals. So let us find $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ and then get the angle between them.

First define

$$\boldsymbol{n}_1 = \overrightarrow{OP} \times \overrightarrow{OR} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= \hat{\boldsymbol{i}}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - \hat{\boldsymbol{j}}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix},$$

$$= 3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}.$$

Its magnitude is $\sqrt{3^2 + (-6)^2} = 3\sqrt{5}$, and so $\hat{n}_1 = (\hat{i} - 2\hat{j})/\sqrt{5}$.

Similarly, we find $n_2 = \overrightarrow{OQ} \times \overrightarrow{OR} = 6\hat{i} - 3\hat{j}$. And that $\hat{n}_2 = (2\hat{i} - \hat{j})/\sqrt{5}$.

The angle $\theta$ between these vectors can be obtained from their dot product:

$$\hat{n}_1 \cdot \hat{n}_2 = \cos\theta,$$

and so $\theta = \cos^{-1}(4/5) \approx 36.9°$.

(b) The normal to the face $PQR$ is $n_3 = \overrightarrow{PQ} \wedge \overrightarrow{PR}$, where

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (0, 0, 3) - (2, 1, 1) = (-2, -1, 2),$$
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (1, 2, 2) - (2, 1, 1) = (-1, 1, 1).$$

We obtain:

$$\overrightarrow{PQ} \wedge \overrightarrow{PR} = n_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ -2 & -1 & 2 \end{vmatrix} = 3\hat{i} + 3\hat{k}$$

We can get the angle $\phi$, once again from the dot product:

$$\overrightarrow{OP} \cdot n_3 = |\overrightarrow{OP}||n_3|\cos\phi,$$
$$(2, 1, 1) \cdot (3, 0, 3) = \sqrt{4 + 1 + 1}\sqrt{3^2 + 3^2}\cos\phi,$$
$$9 = 6\sqrt{3}\cos\phi,$$

and so $\phi = \cos^{-1}(\sqrt{3}/2) = \pi/6$.

(c) The area of the triangle $PQR$ may be expressed using the vector product

$$= \frac{1}{2}|\overrightarrow{PR} \wedge \overrightarrow{PQ}| = \frac{1}{2}|3\hat{i} + 3\hat{k}| = \frac{3\sqrt{2}}{2}.$$

(d) Let us first derive the equation of the plane defined by the triangular surface $PQR$. We require one point on the plane and the normal to the plane. We can

select $\overrightarrow{OP} = (2,1,1) \equiv \boldsymbol{p}$ to be our point on the plane. A normal to the plane is actually just $\boldsymbol{n}_3$ calculated earlier, which we can normalise by $3\sqrt{2}$ to make a unit vector $\hat{\boldsymbol{n}}_3$. Then our equation of the plane is just

$$(\boldsymbol{r} - \boldsymbol{p}) \cdot \hat{\boldsymbol{n}}_3 = 0.$$

From earlier in the notes, another equation for the plane is $\boldsymbol{r} \cdot \hat{\boldsymbol{n}}_3 = d$, where $d$ is the the closest point on the plane to the origin. By comparing the two equations we get

$$d = \boldsymbol{p} \cdot \hat{\boldsymbol{n}}_3 = (2,1,1) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2}) = 3/\sqrt{2},$$

our answer.

Example 1.21 [Tripos question 2003 Paper 1, 1A]

- The vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ form reciprocal sets, defined such that

$$\boldsymbol{A} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}, \quad \boldsymbol{B} = \frac{\boldsymbol{c} \times \boldsymbol{a}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})}, \quad \boldsymbol{C} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})} \quad (84)$$

  and $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \neq 0$. Show that:

  (a) $\boldsymbol{A} \cdot \mathrm{a} = \boldsymbol{B} \cdot \boldsymbol{b} = \boldsymbol{C} \cdot \boldsymbol{c} = 1$

  (b) $\boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{A} \cdot \boldsymbol{c} = 0$

  (c) $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are non-coplanar.

- Demonstrate that the vectors $\boldsymbol{a} = (-1,1,0)$ and $\boldsymbol{b} = (0,2,1)$ and $\boldsymbol{c} = (1,0,-1)$ are non-coplanar. Find its reciprocal basis, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, and hence write the vector $\boldsymbol{d} = (2,1,-1)$ in terms of the basis $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

(a)

$$A \cdot a = \frac{b \wedge c \cdot a}{a \cdot b \wedge c} = \frac{a \cdot b \wedge c}{a \cdot b \wedge c} = 1,$$

$$B \cdot b = \frac{c \wedge a \cdot b}{a \cdot b \wedge c} = \frac{a \cdot b \wedge c}{a \cdot b \wedge c} = 1,$$

$$C \cdot c = \frac{a \wedge b \cdot c}{a \cdot b \wedge c} = \frac{a \cdot b \wedge c}{a \cdot b \wedge c} = 1,$$

where in each case we have cyclically permuted the vectors in the scalar triple product.

(b)

$$A \cdot b = \frac{b \wedge c \cdot b}{a \cdot b \wedge c} = \frac{b \wedge b \cdot c}{a \cdot b \wedge c} = 0,$$

$$A \cdot c = \frac{b \wedge c \cdot c}{a \cdot b \wedge c} = 0$$

(c) If $A$, $B$, and $C$ are non-coplanar then their scalar triple product must be non-zero, i.e. $A \cdot B \wedge C \neq 0$. Let us evaluate the triple product, but simplifying the cross product first:

$$B \wedge C = \frac{(c \wedge a) \wedge (a \wedge b)}{(a \cdot b \wedge c)^2}.$$

Like in a previous worked example, the quadruple product can be simplified if we set $s = c \wedge a$ and use the identity

$$s \wedge (a \wedge b) = (b \cdot s)a - (a \cdot s)b,$$
$$= (b \cdot c \wedge a)a - (a \cdot c \wedge c)b,$$
$$= (a \cdot b \wedge c)a.$$

Thus $B \wedge C = (a \cdot b \wedge c)^{-1} a$. Now let us look at the scalar triple product itself:

$$A \cdot B \wedge C = \frac{(b \wedge c) \cdot a}{(a \cdot b \wedge c)^2},$$

$$= \frac{1}{a \cdot b \wedge c} \neq 0$$

because $a \cdot b \wedge c \neq 0$. Hence $A$, $B$, and $C$ are *not* coplanar.

To show that $a$, $b$, and $c$ are non-coplanar we look at

$$a \cdot b \wedge c = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix},$$

$$= -1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + 0,$$

$$= 2 + 1 = 3 \neq 0$$

so they are non-coplanar.

To compute the reciprocal basis we need some vector products:

$$b \wedge c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-2, 1, -2),$$

$$c \wedge a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} = (1, 1, 1),$$

$$a \wedge b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = (1, 1, -2),$$

And since their scalar triple product is 3, we have

$$A = \tfrac{1}{3}(-2, 1, -2), \quad B = \tfrac{1}{3}(1, 1, 1), \quad C = \tfrac{1}{3}(1, 1, -2)$$

Now to write $d$ in terms of the *non-orthogonal* basis $a$, $b$, $c$ we could set

$$d = \alpha a + \beta b + \gamma c$$

where $\alpha$, $\beta$, and $\gamma$ are constant components that we need to find. The $x$, $y$, and $z$ coordinates of this vector equation yield three scalar equations for three unknowns, and it's easy to solve:

$$2 = -\alpha + \gamma, \quad 1 = \alpha + 2\beta, \quad -1 = \beta - \gamma,$$

for $\alpha$, $\beta$, and $\gamma$.

A better way, using the reciprocal basis, is to dot both sides of the equation with $\boldsymbol{A}$. Using our previous results we obtain

$$\alpha = \boldsymbol{d} \cdot \boldsymbol{A} = (2, 1, -1) \cdot \tfrac{1}{3}(-2, 1, -2) = -\tfrac{1}{3},$$

similarly

$$\beta = \boldsymbol{d} \cdot \boldsymbol{B} = (2, 1, -1) \cdot \tfrac{1}{3}(1, 1, 1) = \tfrac{2}{3},$$
$$\gamma = \boldsymbol{d} \cdot \boldsymbol{C} = (2, 1, -1) \cdot \tfrac{1}{3}(1, 1, -2) = \tfrac{5}{3}.$$

Thus

$$\boldsymbol{d} = -\tfrac{1}{3}\boldsymbol{a} + \tfrac{2}{3}\boldsymbol{b} + \tfrac{5}{3}\boldsymbol{c}.$$