Natural Sciences Tripos, Part IA
Mathematical Methods II, Course A
Examples Sheet 3
Multiple integrals and fields

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This examples sheet is for course A. Please communicate any errors to `mgw1@cam.ac.uk`.

Questions intended to give practice with routine calculations are signified by a leading 'S', and answers to these questions are given at the end of this sheet. Starred questions are more challenging and might be left for later use when more experienced.

Multiple integrals

S1. Evaluate the following integrals:

(a) $\int_0^1 \int_0^1 (x + y + 1)^{-2}\, dxdy$

(b) (without changing variables) $\int_{\mathcal{D}} xy\, dxdy$

(c) (changing variables to 2D polars) $\int_{\mathcal{D}} xy\, dxdy$

(d) (changing variables to 2D polars) $\int_{\mathcal{D}} dxdy$

In (b), (c) and (d) $\mathcal{D}$ is the quadrant of the circle of radius 1 and centred on the origin which lies in $x \geq 0$, $y \geq 0$.

2. Evaluate the integral

$$\int_{x=0}^{2} \int_{y=x/2}^{1} 2xy^2 \; dy \; dx.$$

Evaluate the integral again by first changing the order of integration (so integrating with respect to x first). Verify that the second answer is the same as the first.

3. Sketch the curve given by $r = 2a(1 + \cos\theta)$ and evaluate
$$\int_{\mathcal{D}} (x^2 + y^2)^{1/2} dxdy,$$
where $\mathcal{D}$ is the interior of the curve.

4. Evaluate the integral $\int x^2(1-x^2-y^2)\,dxdy$ over the interior of the circle of radius 1 centred at $x = 0, \quad y = 0$.

5. Show that
$$\int_{\mathcal{D}} x^2 y dxdy = (32\sqrt{2} - 11)/42,$$
where $\mathcal{D}$ is the smaller of the two areas bounded by the curves $xy = 1$, $y = x^2$ and $y = 2$.

6. By evaluating the appropriate triple integrals find

   (a) the volume of a sphere of radius a;

   (b) the volume of a flat-topped cone, described, in cylindrical polar coordinates by $\{(r, \theta, z)|0 \leq z \leq a,\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq z\}$;

   (c) the volume of a round-topped cone, described, in spherical polar coordinates by $\{(r, \theta, \phi)|0 \leq r \leq a,\ 0 \leq \phi \leq 2\pi,\ 0 \leq \theta \leq \frac{\pi}{4}\}$.

7. Evaluate the volume integral of $e^{-(x^2+y^2+z^2)/a^2}$ over the whole of three-dimensional space.

8*. By means of mathematical induction show that
$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n-1)(2n-3)...3 \times 1}{2(2a)^n} \left\{\frac{\pi}{a}\right\}^{1/2}$$
for integer n and positive a.

9*. According to the Schrödinger equation of quantum mechanics, an electron in the ground state of a hydrogen atom has a spherically symmetric spatial probability distribution $P(r) = Ke^{-2r/a}$, i.e. the probability of finding an electron in a volume dV at radius r is $P(r)dV$. Here r is the distance of the electron from the proton, a is a fixed constant with dimension of length and K is a (dimensional) constant chosen to make the probability of finding the electron somewhere in the whole of space equal to unity. Calculate:

   (i) the constant K;

   (ii) the mean value of r;

   (iii) the most probable value of r.

Scalar and vector fields

S10. For each of the following functions $f(x, y, z)$ or $f(\mathbf{x})$, evaluate ∇f.

(a) $f = xyz$

(b) $f = \exp(-1/(x + y + z))$

(c) $f = \exp(-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2)$

(d) $f = r$

(e) $f = r^{-1}$

(f) $f = G(r)$, for an arbitrary function G.

[In (d), (e) and (f), $r = |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2}$]

S11. Write down parametrizations of the following curves:

(a) circle in xy plane, centre at $(1, 1, 0)$ and passing through the origin, traversed clockwise starting from $(0, 0, 0)$;

(b) straight line from $(-1, 0, 0)$ to $(1, 1, 1)$;

(c) triangle with corners $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

S12. Sketch each of the following vector fields and compute their divergence and curl.

(a) $\mathbf{F}(\mathbf{x}) = (x, y)$

(b) $\mathbf{F}(\mathbf{x}) = (-y, x)$

(c) $\mathbf{F}(\mathbf{x}) = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}\right)$

(d) $\mathbf{F}(\mathbf{x}) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}\right)$

For the vector field in (c), find a scalar field $\phi(x, y)$ such that $\mathbf{F} = \nabla\phi$.

S13. Let $\mathbf{a}$ and $\mathbf{b}$ be fixed vectors.

(a) Show that $\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$.

(b) Evaluate the divergence and the curl of each of the following vector fields:

$$\mathbf{x}; \qquad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}); \qquad \mathbf{a} \times \mathbf{x}; \qquad \mathbf{x}/r^3, \text{ where } r = |\mathbf{x}| \ .$$

14. For the function $f(x, y, z) = \ln(x^2 + y^2) + z$, find ∇f.

Consider a cylinder of radius 5 whose axis is along the z-axis.

(i) What is the rate of change of $f(x, y, z)$ in the direction normal to the cylinder at the point $(3, -4, 4)$?

(ii) What is the rate of change of $f(x, y, z)$ in the direction $\mathbf{m} = \mathbf{i} + 2\mathbf{j}$ at the same point, where $\mathbf{i}, \mathbf{j}$ are unit vectors parallel to the x, y axes respectively.

15. Find the normal to the surface $xz + z^2 - xy^2 = 5$ at the point (1,1,2). Deduce the equation of the tangent plane at this point.

16. Obtain the equation of the plane that is tangent to the surface $z = 3x^2 y \sin(\pi x/2)$ at the point $x = y = 1$. Take North to be the direction $(0, 1, 0)$ and East to be the direction $(1, 0, 0)$. In which direction will a marble roll if placed on the surface at $x = 1, y = \frac{1}{2}$?

17. Evaluate
$$\int_\Gamma \{P(x,y)dx + Q(x,y)dy\},$$
where $P = -x^2 y, \quad Q = y^2 x$ and Γ is the closed curve consisting of the semi-circle $x^2 + y^2 = a^2$, $(y > 0)$, and the segment $(-a, a)$ of the x-axis, described anti-clockwise. Verify that this is equal to
$$\int_{\mathcal{D}} \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dxdy,$$
where $\mathcal{D}$ is the plane surface enclosed by Γ. (This illustrates Stokes's Theorem in the plane.)

18. Determine a function $f(y)$ such that
$$\int_\Gamma [f(y)\, dx + x \cos y\, dy] = 0$$
for all closed contours Γ in the (x, y) plane.

[*Hint: when is the integrand an exact differential?*]

19. The work done by a force $\mathbf{F}$ acting on a particle which moves along a curve C is defined as the line integral
$$W = \int_C \mathbf{F} \cdot \mathbf{dx}.$$

(i) When $\mathbf{F} = \mathbf{c} \times \mathbf{v}$, where $\mathbf{c}$ is a constant vector, and $\mathbf{v} = \dfrac{d\mathbf{x}}{dt}$ is the velocity of the particle, show that the work done is zero.

(ii) A particle moves along the helical path given by
$$x = \cos t, \quad y = \sin t, \quad z = t.$$

Calculate the work W done in the time interval $0 \le t \le \pi$ by each of the forces $\mathbf{F}$ given by

(a) $y\mathbf{i} - x\mathbf{j} - \mathbf{k}$,

(b) $x\mathbf{i} + y\mathbf{j}$,

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel to the x, y, z axes respectively.

20. Write down a condition obeyed by a conservative vector field $\mathbf{V} = \{P(x,y),\ Q(x,y)\}$. Do the following choices for $P,\ Q$ yield conservative fields ?

(i) $P = x^2y + y$, $Q = xy^2 + x$;

(ii) $P = ye^{xy} + 2x + y$, $Q = xe^{xy} + x$.

In the case that $\mathbf{V}$ is conservative, find a function $f(x,y)$ such that $\mathbf{V} = \nabla f$.

Consider the following curves, each joining $(0,0)$ to $(1,1)$:

(a) $C_1 : x = t, \quad y = t \quad (0 \le t \le 1)$

(b) $C_2 : x = 0, \quad y = t \quad (0 \le t \le 1); \quad x = t, \quad y = 1 \quad (0 \le t \le 1)$.

Evaluate the integrals $\displaystyle\int_{C_1} Pdx + Qdy$ and $\displaystyle\int_{C_2} Pdx + Qdy$ for each of the fields (i) and (ii) above.

21. Consider the vector field

$$\mathbf{V} = (4x^3z + 2x, \quad z^2 - 2y, \quad x^4 + 2yz).$$

Evaluate the line integral $\displaystyle\int_C \mathbf{V} \cdot \mathbf{ds}$ along

(i) the sequence of straight-line paths joining $(0,0,0)$ to $(0,0,1)$ to $(0,1,1)$ to $(1,1,1)$.

(ii) the straight line joining $(0,0,0)$ to $(1,1,1)$, given parametrically by

$$x = y = z = t \qquad (0 \le t \le 1).$$

Show that $\mathbf{V}$ is conservative by finding a function $f(x,y,z)$ such that $\mathbf{V} = \nabla f$.

22. Let $\mathbf{E} = (-ye^{-2t}, \quad xe^{-2t}, \quad 0)$ and $\mathbf{B} = (0, 0, e^{-2t})$. Evaluate $\displaystyle\int_S \mathbf{B} \cdot \mathbf{dS}$, $\displaystyle\int_C \mathbf{E} \cdot \mathbf{dx}$, where S is the surface of the circular disc $x^2 + y^2 < 1, \quad z = 0$ and C is the curve bounding S. Show that

$$\int_C \mathbf{E} \cdot \mathbf{dx} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}.$$

23. Calculate $\displaystyle\int \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \alpha x^3 \mathbf{i} + \beta y^3 \mathbf{j} + \gamma z^3 \mathbf{k}$ (with α, β and γ constants):

(i) over the surface of a sphere of radius a, centred at the origin.

(ii) over the surface of a cylinder of radius a and height $2h$, centred at the origin with its axis in the z-direction.

[Hint: You can reduce the amount of algebra by exploiting the symmetry of the sphere and the cylinder.]

24. A cube is defined by $0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1$. Evaluate the surface integral

$$\int \mathbf{F} \cdot \mathbf{n}\, dS$$

over the surface of the cube where $\mathbf{F} = (x^2 + ay^2,\ 3xy,\ 6z)$ and a is a constant. [$\mathbf{n}$ is a unit vector in the direction of the outward normal from the volume across a surface element dS; e.g. on the $x = 0$ face, $dS = dydz$ and $\mathbf{n} = (-1, 0, 0)$.]

Evaluate also

$$\int f dV$$

over the volume of the same cube, where $f = bx + 6$ and b is a constant. For what values of a and b do these integrals have the same value?

25. Let $\mathbf{u}$ be the vector field $\mathbf{u} = Q\mathbf{x}/4\pi\epsilon_0 r^3$ in three dimensions, where $\mathbf{x}$ is the position vector and $r = |\mathbf{x}|$ (Q, ϵ_0 constants). Show that

$$\int_S \mathbf{u} \cdot \mathbf{n}\, dS = Q/\epsilon_0 \quad ,$$

where S is a sphere of radius a centred on the origin, and $\mathbf{n}$ is a unit normal vector pointing radially outward from the sphere. [This is Gauss's law for a point charge.]

Answers to Skills questions

S1. (a) $\ln \frac{4}{3}$ (b) $\frac{1}{8}$ (c) $\frac{1}{8}$ (d) $\frac{\pi}{4}$

S10 (a) (yz, zx, xy)

(b) $\dfrac{\exp(-1/(x+y+z))}{(x+y+z)^2}(1, 1, 1)$

(c) $\exp(-\alpha^2 x^2 - \beta^2 y^2 - \gamma^2 z^2)(-2\alpha^2 x, -2\beta^2 y, -2\gamma^2 z)$

(d) $\left(\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right)$

(e) $\left(-\dfrac{x}{r^3}, \dfrac{-y}{r^3}, \dfrac{-z}{r^3}\right)$

S11 (a) $x = 1 + \sqrt{2}\cos t,\ y = 1 + \sqrt{2}\sin t,\ z = 0,\ t : \frac{5}{4}\pi \to -\frac{3}{4}\pi$

(b) $x = -1 + 2t,\ y = t,\ z = t,\ 0 < t < 1$

(c) $x = 1 - r,\ y = r,\ z = 0,\ 0 < r < 1$ followed by $x = 0,\ y = 1 - s,\ z = s,\ 0 < s < 1$ followed by $x = t,\ y = 0,\ z = 1 - t,\ 0 < t < 1$

S12 (a) $\nabla \cdot \mathbf{F} = 2$, $\nabla \times \mathbf{F} = (0, 0, 0)$

(b) $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = (0, 0, 2)$

(c) $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = (0, 0, 0)$, except at the origin

(d) $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = (0, 0, 0)$, except at the origin

$\phi = \frac{1}{2} \ln(x^2 + y^2) = \ln r$

S13 (b) 3, $\mathbf{0}$; $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \times \mathbf{a}$; 0, $2\mathbf{a}$; 0, $\mathbf{0}$, except at the origin