Natural Sciences Tripos, Part IA

Mathematical Methods II, Course A

Examples Sheet 2
Functions of More than One Variable

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This examples sheet is for course A. Please communicate any errors to `mgw1@cam.ac.uk`.

Skills section

Questions in this section are intended to give practice in routine calculations. You should not need to spend much supervision time on these. Answers are given at the end of this sheet.

S1. For each of the following functions, evaluate the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}$, and verify that $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$.

(a) $f = x^3 - 3x^2y + 3xy^2 + 8y^3 - 3y$ (b) $f = \exp(-x^2y^2)$

(c) $f = 1/(x^2 + xy + 2y^2)$

S2. For each of the following functions, write out the differential expression $\mathrm{d}f = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ showing the explicit form of the functions $P(x,y)$ and $Q(x,y)$.

(a) $f = \exp[-1/(x+y)]$ (b) $f = \sinh x/\sinh y$

(c) $f = (x^2 + y^2)^{1/2}$ (d) $f = \arctan(y/x)$

(e) $f = x^y$

S3. For the functions in question S1(a) and S1(b), find the locations of any stationary points. (You need not determine the type of the stationary points.)

Standard questions

Partial differentiation

4. For the function
$$f(x,y) = x(y^2 + 2y - 1)\,,$$
find the components of the vector $(\partial f/\partial x, \partial f/\partial y)$, known as the gradient vector, at the points $(-1,0)$, $(1,0)$, $(-1,1)$ and $(1,1)$. Make a sketch showing the directions of the gradient vector at these points.

5. The period T of a simple pendulum of length ℓ swinging in a gravitational field g is given by
$$T = 2\pi(\ell/g)^{1/2}\,.$$
By finding the differential of $\ln T$, estimate the percentage error in a measurement of g resulting from

   (a) a 0.1 % error in measuring ℓ,

   (b) a 0.1 % error in measuring T.

6. For $f(x,y) = \exp(-xy)$, find $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$. Then find $(\partial f/\partial r)_\theta$ and $(\partial f/\partial \theta)_r$ if (x,y) are Cartesian coordinates and (r,θ) are the corresponding polar coordinates,

   (i) by using the chain rule,

   (ii) by first expressing f in terms of (r,θ),

   and check that the two methods give the same results.

7. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find
$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y,$$
and show that their product is -1.

8. $f(x,y)$ is a scalar function of position on the (x,y) plane. Position may also be specified by Cartesian coordinates u, v which are referred to axes rotated by a constant angle θ from the x and y axes. First, with the aid of a diagram or otherwise, show that
$$x = u\cos\theta - v\sin\theta,$$
$$y = u\sin\theta + v\cos\theta.$$
Then show that
$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,,$$
i.e. the two-dimensional ∇^2 operator is invariant under a rotation of axes.

Exact differentials

9. Determine whether the following differentials $P\,\mathrm{d}x + Q\,\mathrm{d}y$ are exact. For those that are exact, find a function f such that $\mathrm{d}f = P\,\mathrm{d}x + Q\,\mathrm{d}y$. In each case, find the general solution of the differential equation $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$.

   (a) $y\,\mathrm{d}x + x\,\mathrm{d}y$

   (b) $y\,\mathrm{d}x + x^2\,\mathrm{d}y$

   (c) $(x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y$

   (d) $(\cosh x \cos y + \cosh y \cos x)\,\mathrm{d}x - (\sinh x \sin y - \sinh y \sin x)\,\mathrm{d}y$

   (e) $(\cos x - \sin x)\,\mathrm{d}x + (\sin x + \cos x)\,\mathrm{d}y$

   (f) $(x\,\mathrm{d}y - y\,\mathrm{d}x)/(x^2+y^2)$

   (You may wish to discuss with your supervisor why case (f) presents difficulties.)

10. The enthalpy of a gas is defined by $H = U + pV$, where U satisfies $\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V$. Determine a relationship between the differentials of H, S and p. Hence show that

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S .$$

   By regarding U as a function of p and V and considering two expressions for $\partial^2 U/\partial p\,\partial V$, show that

$$\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial T}{\partial p}\right)_V - \left(\frac{\partial S}{\partial p}\right)_V \left(\frac{\partial T}{\partial V}\right)_p = 1 .$$

11. Given that

$$\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V ,$$

   find a function G such that

$$\mathrm{d}G = V\,\mathrm{d}p - S\,\mathrm{d}T .$$

   Hence show that

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

12. The pressure p can be considered as a function of the variables V and T or as a function of the variables V and S. Starting from the differential expression of the chain rule for $p(V, S)$,

(i) Find an expression for

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \quad \text{in terms of} \quad \left(\frac{\partial S}{\partial V}\right)_T \quad \text{and} \quad \left(\frac{\partial S}{\partial p}\right)_V .$$

(ii) Hence, using $T\,\mathrm{d}S = \mathrm{d}U + p\,\mathrm{d}V$, show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S = \left(\frac{\partial (pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right] . \qquad (\star)$$

$\left[\textit{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)_T .\right]$

(iii) For one mole of an ideal gas, you may assume that $U = C_\mathrm{v} T$, $pV = RT$, and pV^γ depends only on S, where C_v, R and γ are constants.
Use equation $(\star)$ to find an expression for γ in terms of C_v and R.

(iv) What is the value of γ for a monatomic gas for which $C_\mathrm{v} = \frac{3}{2}R$?

Stationary values

13. The height h of each point (x, y) of an area of land is given by

$$h(x, y) = \frac{a(x + y)}{x^2 + y^2 + a^2} ,$$

where a is a positive constant. Sketch a map of the region by showing contours of constant h in the (x, y) plane and identifying the locations and heights of the highest and lowest points of the terrain. Determine also the locations of the highest and lowest points along the x and y axes.

14. (i) Find the stationary points of the function

$$z = (x^2 - y^2)\,\mathrm{e}^{-x^2-y^2} .$$

(ii) Find the contours on which $z = 0$ and examine the behaviour of z on the x and y axes. Hence, or otherwise, determine the character of the stationary points. Sketch the contours.

15. For each of the functions

(a) $(x^2 + y^2 + 1)^{-1}$

(b) $\sin x \sin y \quad (0 < x < \pi,\ 0 < y < \pi)$

(c) $2x^3 + 6xy^2 - 3y^3 - 150x$

find the stationary points and determine their character.

Partial differential equations

16. Show that the following functions satisfy Laplace's

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \ .$$

(i) $\psi = x^4 - 6x^2y^2 + y^4$.

(ii) $\psi = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y$.

(iii) $\psi = \ln\{[x^2 + (y-1)^2]/[x^2 + (y+1)^2]\}$ (except at $x = 0$, $y = 1$, and $x = 0$, $y = -1$).

In (i) show also that $\partial\psi/\partial y = 0$ on $y = 0$ and $\partial\psi/\partial y - \partial\psi/\partial x = 0$ on $y = x$. In (iii) show also that $\psi = 0$ on $y = 0$. (These are boundary conditions that are relevant in particular physical problems.)

17. A function $\psi(x, y)$ is defined in the strip $x \geq 0,\ 0 \leq y \leq a$ and satisfies Laplace's equation, namely

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \ .$$

$\psi(x, y) \to 0$ as $x \to \infty$, and satisfies the boundary conditions

$$\begin{aligned} \psi(x,0) &= \psi(x,a) = 0 \ , \\ \psi(0,y) &= \sin\frac{\pi y}{a} + 2\sin\frac{2\pi y}{a} \ . \end{aligned}$$

Show that

$$\psi(x,y) = \sin\frac{\pi y}{a} e^{-\pi x/a} + 2\sin\frac{2\pi y}{a} e^{-2\pi x/a}$$

satisfies the equation and the boundary conditions.

18. N drunkards emerge at closing time from a pub and diffuse throughout the town. Given that the density $u(r,t)$, of the drunks at distance r from the pub at time t obeys the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{2}{\beta}\frac{\partial u}{\partial t} \ ,$$

where β is constant, verify that a suitable solution is

$$u(r,t) = \frac{\alpha}{t}\exp\left(\frac{-r^2}{2\beta t}\right)$$

when α is constant. The total number of drunkards, N is equal to $2\pi\int_0^\infty u(r,t)\, r\, dr$. (Why?) Show that $N = 2\pi\alpha\beta$, and show further that the density of drunkards at distance R from the pub never exceeds $N/(R^2\pi e)$.

19. The transverse displacement $u(x,t)$ of a string obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ ,$$

where t and x denote time and position respectively and c is a constant. Show that a possible solution is given by

$$u(x,t) = f(x-ct) + g(x+ct) \ ,$$

where f and g are any functions that can be differentiated twice.

Suppose that the string stretches to infinity in both directions, and satisfies the intial conditions,

$$u(x,0) = 0, \qquad \partial u/\partial t(x,0) = \frac{x}{(1+x^2)^2}.$$

Show that the initial conditions can be satisfied by taking $f = -g$ for suitable f. Find the solution $u(x,t)$ and sketch its behaviour.

Answers to Skills questions

S1 (a)
$f_x = 3x^2 - 6xy + 3y^2$,
$f_y = -3x^2 + 6xy + 24y^2 - 3,$
$f_{xx} = 6x - 6y,$
$f_{yy} = 6x + 48y,$
$f_{xy} = -6x + 6y.$

(b)
$f_x = -2xy^2 e^{-x^2y^2},$
$f_y = -2x^2 y e^{-x^2y^2},$
$f_{xx} = (-2y^2 + 4x^2y^4)e^{-x^2y^2},$
$f_{yy} = (-2x^2 + 4x^4y^2)e^{-x^2y^2},$
$f_{xy} = (-4xy + 4x^3y^3)e^{-x^2y^2}.$

(c)
$f_x = -(2x + y)(x^2 + xy + 2y^2)^{-2},$
$f_y = -(x + 4y)(x^2 + xy + 2y^2)^{-2},$
$f_{xx} = (6x^2 + 6xy - 2y^2)(x^2 + xy + 2y^2)^{-3},$
$f_{yy} = (-2x^2 + 12xy + 24y^2)(x^2 + xy + 2y^2)^{-3},$
$f_{xy} = (3x^2 + 17xy + 6y^2)(x^2 + xy + 2y^2)^{-3}.$

S2 (a) $df = \dfrac{e^{-1/(x+y)}}{(x+y)^2}(dx + dy)$

(b) $df = \dfrac{\cosh x \sinh y \, dx - \sinh x \cosh y \, dy}{\sinh^2 y}$

(c) $df = \dfrac{x\, dx + y\, dy}{(x^2 + y^2)^{1/2}}$

(d) $df = \dfrac{-y\, dx + x\, dy}{x^2 + y^2}$

(e) $df = yx^{y-1}\, dx + (\ln x)x^y\, dy$

S3 (a) Stationary points at $(\frac{1}{3}, \frac{1}{3})$ and $(-\frac{1}{3}, -\frac{1}{3})$

(b) Stationary points at $(0, 0)$ and along x and y axes