Natural Sciences Part IA
Mathematics II (Course A)

Examples Sheet 1
Ordinary Differential Equations

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This examples sheet is for Course A. Please communicate any errors to mgw1@cam.ac.uk

Basic skills

Questions intended to give practice with routine calculations are signified by a leading 'S', and answers to these questions are given at the end of this sheet. On this sheet, the skills questions start off with integration and differentiation, as revision from Michaelmas Term, since these are important mathematical operations in the solution of ordinary differential equations. Answers to the Skills questions are given at the end of this sheet.

S1 Find the indefinite integral $\int f(x)\,dx$ when $f(x)$ is given by:

(a) $(1+x)^{1/4}$

(b) $\dfrac{2+x}{(1+x)^2}$ where $x \neq -1$

(c) $2x(3+x^2)$

(d) $2x\sin(x^2)$

(e) $\cot x$

(f) $x^2 \sin x$ [*Hint: integration by parts*]

(g) $\sin^3 x$ [*Hint: trig. identity*]

(h) $\dfrac{x}{(1-x)(2-x)}$ with $x \neq 1, 2$ [*Hint: partial fractions*]

(i) $\dfrac{1}{1+x^2}$ [*Hint: substitution*]

(j) $\sin\sqrt{1-x}$ [*Hint: substitution*]

S2 Verify that the following are solutions of the corresponding differential equations, where c and d are arbitrary constants.

(a) $\dfrac{dy}{dx} = x$ $\qquad y = \frac{1}{2}x^2 + c$

(b) $\dfrac{dy}{dx} = y$ $\qquad y = ce^x$

(c) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 0$ $\qquad y = ce^{-4x} + de^{-x}$

(d) $x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} + 4y = 0$ $\qquad y = cx^{-4} + dx^{-1}$

(e) $\left(\dfrac{dy}{dx}\right)^2 + 4y^3 = \dfrac{8}{x^6}$ $\qquad y = x^{-2}$ [Note: This is not the general solution.]

S3 Solve by separation of variables

(a)
$$\frac{dy}{dx} = -\frac{x^3}{(y+1)^2}$$

(b)
$$\frac{dy}{dx} = \frac{-4y}{x(y-3)}$$

S4 Solve by use of integrating factors

(a)
$$\frac{dy}{dx} + 2xy = 4x$$

(b)
$$\frac{dy}{dx} + (2 - 3x^2)x^{-3}y = 1$$

5. Find the general solutions of the following differential equations and then determine the solutions that satisfy the given boundary conditions. First identify the type of equation and use the simplest method that applies.

(a)
$$\frac{dy}{dx} = \frac{y^2+1}{\cos^2 x}, \qquad y(0) = 0$$

(b)
$$\frac{dy}{dx} + 4xy = 2x(y^2+1), \qquad y(0) = 0$$

(c)
$$x\frac{dy}{dx} + (x-1)y = e^{-x}, \qquad y(1) = 0$$

(d)
$$(1+x^3)\frac{dy}{dx} - x^2y = x^2, \qquad y(0) = 0$$

6. Solve the homogeneous equation

$$(y-x)\frac{dy}{dx}+(2x+3y)=0.$$

7. Solve the Bernoulli equation

$$\frac{dy}{dx}-y=xy^5.$$

8. Find suitable substitutions and solve the equations

(a)
$$\frac{dy}{dx}+y=y^2(\cos x-\sin x)$$

(b)
$$\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$$

(c)
$$(\ln y-x)\frac{dy}{dx}-y\ln y=0$$

(d)
$$xy\frac{dy}{dx}+(x^2+y^2+x)=0$$

S9. Find the general solution (complementary function) of the following homogeneous (unforced) differential equations

(a)
$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

(b)
$$2\frac{d^2y}{dx^2}-3\frac{dy}{dx}-2y=0$$

(c)
$$\frac{d^2y}{dx^2}-9y=0$$

(d)
$$\frac{d^2y}{dx^2}+4y=0$$

(e)
$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=0$$

S10. Find particular integrals for the following differential equations

(a)
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x$$

(b)
$$\frac{d^2y}{dx^2} - 9y = \exp 2x$$

(c)
$$\frac{d^2y}{dx^2} - 9y = \exp 3x$$

(d)
$$\frac{d^2y}{dx^2} - 9y = 2\exp 2x + \exp 3x$$

(e)
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$$

11. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2x + 3\sin x.$$

12. Solve the following differential equations subject to the boundary conditions $y(0) = 0,\ \frac{dy}{dx}(0) = 1.$

(a)
$$\frac{d^2y}{dx^2} - n^2y = 0$$

(b)
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

(c)
$$\frac{d^2y}{dx^2} + n^2y = 0$$

(d)
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

(e)
$$\frac{d^2y}{dx^2} + 9y = \sin 3x$$

(f)
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$$

13. (a) Solve the pair of first-order differential equations

$$\frac{dx}{dt} = ax, \qquad \frac{dy}{dt} = ay + bx$$

sequentially, where a and b are constants, subject to the initial conditions $x(0) = 2,\ y(0) = 1$.

(b) Combine the equations above to show that $y(t)$ satisfies the second-order differential equation

$$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + a^2y = 0,$$

subject to the initial conditions $y(0) = 1,\ \dfrac{dy}{dt}(0) = a + 2b$. Solve this second-order system and verify that it gives the same solution as that found in part (a).

Answers to Skills questions

S1 (a) $\frac{4}{5}(1+x)^{5/4} + c$

(b) $\ln(1+x) - \dfrac{1}{1+x} + c$

(c) $\frac{1}{2}(3+x^2)^2 + c$

(d) $-\cos(x^2) + c$

(e) $\ln(\sin x) + c$

(f) $-x^2\cos x + 2x\sin x + 2\cos x + c$

(g) $-\cos x + \frac{1}{3}\cos^3 x + c$

(h) $-\ln(1-x) + 2\ln(2-x) + c = \ln\frac{(2-x)^2}{1-x} + c$

(i) $\tan^{-1} x + c$

(j) Let $u = \sqrt{1-x}$. Then

$$\int \sin\sqrt{1-x}\,dx = 2u\cos u - 2\sin u + c = 2\sqrt{1-x}\cos\sqrt{1-x} - 2\sin\sqrt{1-x} + c$$

S3 (a) $y = \left(c - \dfrac{3x^4}{4}\right)^{1/3} - 1$

(b) $\dfrac{e^y}{y^3} = \dfrac{A}{x^4}$

S4 (a) $y = 2 + ce^{-x^2}$

(b) $y = \dfrac{x^3}{2} + cx^3\exp\left(\dfrac{1}{x^2}\right)$

S9 In the answers below, A, B, α and β are arbitrary constants.

(a) $y = Ae^{2x} + Be^{3x}$

(b) $y = Ae^{-x/2} + Be^{2x}$

(c) $y = Ae^{3x} + Be^{-3x}$ or $y = \alpha \cosh 3x + \beta \sinh 3x$

(d) $y = Ae^{2ix} + Be^{-2ix}$ or $y = \alpha \cos 2x + \beta \sin 2x$

(e) $y = e^{-x}(Ae^{2ix} + Be^{-2ix})$ or $y = e^{-x}(\alpha \cos 2x + \beta \sin 2x)$

S10 Note that you are only asked for a (single) particular integral. Any complementary function may be added to give a different particular integral but is not required.

(a) $y = \frac{1}{36}(6x + 5)$

(b) $y = -\frac{1}{5}e^{2x}$

(c) $y = \frac{1}{6}xe^{3x}$

(d) $y = \frac{1}{6}xe^{3x} - \frac{2}{5}e^{2x}$ Use linear superposition from the previous two questions.

(e) $y = \frac{1}{10}(\cos x + \sin x)$.