Example Sheet II

Mathematical Methods III:
Course A Easter Term 2020

Natural Sciences 1A, Computer Science 1A

Professor Fernando Quevedo

Please, send any corrections or comments by email to f.quevedo@damtp.cam.ac.uk

This sheet provides exercises covering the material contained in the last **5 or 6** lectures of the Easter Term. **ALL** questions should be attempted by students attending Course A lectures.

Students should attempt questions as soon as the work has been covered in lectures - do not wait for your supervisors to set questions.

These questions can be checked using Mathematica or Matlab and students are strongly encouraged to do so. Questions should be done manually first and then checked using Mathematica or Matlab.

1 Fourier Series

1. Show that the functions:

$$\text{(a) } 1 \qquad (b)\ x \qquad \text{(c) } \frac{1}{2}(3x^2-1) \qquad \text{and} \qquad \text{(d) } \frac{1}{2}(5\,x^3-3\,x)$$

are orthogonal on the interval $[-1,1]$. For the measure you can take $W(x)=1$.

2. Without resorting to integration, write down the Fourier series on $[-\pi,\pi]$ (with period $2\,\pi$) for:

   (a) $\sin(2\theta)$

   (b) $\cos^2\theta$

   (c) $\sin^3\theta$

3. Say whether each one of the following functions of x are even, odd or neither:

$$\text{(a) } \cos x \quad \text{(b) } \sin x \quad \text{(c) } \tan x \quad \text{(d) } \cos^2 x \quad \text{(e) } \sin^2 x \quad \text{(f) } x\cos x$$

$$\text{(g) } e^x \quad \text{(h) } \frac{x-1}{x+1}\,.$$

   (i) Given an arbitrary function $f(x)$, show that

$$F(x)=\frac{1}{2}\left[f(x)+f(-x)\right]\,,\quad G(x)=\frac{1}{2}\left[f(x)-f(-x)\right]\,,$$

   are respectively even and odd. Thus

$$f(x)=F(x)+G(x)$$

   is a resolution of $f(x)$ into its even and odd parts.

   (ii) Perform this resolution for the functions in the previous list for which the answer 'neither' was given.

4. A function $g(x)$ of period 2 is defined for $-1<x<1$ by $g(x)=x+|x|$, and by periodicity for all other x.

   (a) Sketch this function, and

   (b) Without resorting to integration, instead using results given in lectures, *write down* its Fourier series.

5. Prove by integrating by parts that:

$$\int_0^1 x^2\cos(n\pi x)\mathrm{d}x=\frac{2(-1)^n}{n^2\pi^2}\,.$$

6. An even function is defined by $f(x)=x^2$ for $-1\le x\le 1$, and by periodicity elsewhere.

   (a) Use the result of the previous question to show that it has the Fourier series

$$f(x)=\frac{1}{3}+\sum_{n=1}^{+\infty}\frac{4(-1)^n}{n^2\pi^2}\cos(n\,\pi\,x)\,.$$

   (b) Show (by setting $x=1$) that

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\,.$$

7. Let

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin(n\pi x) \quad \text{and} \quad g(x) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x)\,.$$

   (a) show that

$$\int_{-1}^{1} f(x)g(x)\mathrm{d}x = \sum_{n=1}^{+\infty} b_n\, B_n\,.$$

   (b) What is the corresponding result when

$$f(x) = g(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(n\pi x)\,.$$

8. If $f(x) = g(x) = x^2$, combine the results of question six and the last result of question seven to deduce the result

$$\frac{\pi^4}{90} = \sum_{n=1}^{+\infty} \frac{1}{n^4}\,.$$

9. A function is defined by $f(x) = \cos x$ for $0 < x < \pi$, by $f(x) = -\cos x$ for $-\pi \leq x \leq 0$, and by periodicity elsewhere.

   (a) Sketch this odd function carefully

   (b) Show that is has the Fourier sine series

$$f(x) = \sum_{n\in\ \text{even}}^{+\infty} \frac{4n}{\pi(n^2-1)} \sin(n\,x) = \sum_{r=1}^{+\infty} \frac{8r}{\pi(4\,r^2-1)} \sin(2\,r\,x)$$

   (c) How do you interpret this result for $x = 0$?

   (This question has found a half-range sine series for the function $f(x) = \cos x$ defined for $0 \leq x \leq \pi$.)