Methods Example Sheet 4

1. (a) Find the characteristic curves of $u_x + yu_y = 0$. Hence find the solution of the problem with the boundary data $u(0, y) = g(y) = y^3$.

   (b) Solve for u the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$. Is the solution defined on the whole xy plane?

   (c) Find u such that $u_x + u_y + u = e^{x+2y}$ and $u(x, 0) = 0$.

2. Define the concept of well-posedness.

   The backwards diffusion equation is $u_{xx} + u_t = 0$. Consider a domain $0 < x < \pi$, with $u(0, t) = u(\pi, t) = 0$ and $u(x, 0) = U(x)$. By using separation of variables and considering a similar problem with initial condition

$$u(x, 0) = U_n(x) \coloneqq U(x) + c\frac{\sin(nx)}{n},$$

   for $n \in \mathbb{Z}$, c a constant, show that the problem is not well-posed.

   Would the result be different if we used the forward (usual) diffusion equation $u_{xx} = u_t$ instead?

3. *Classification.*

   (a) Determine the regions where Tricomi's equation $u_{xx} + xu_{yy} = 0$, is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.

   (b) Reduce the equation

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0,$$

   to the simple canonical form $u_{\xi\eta} = 0$ in its hyperbolic region, and hence show that

$$u = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y}),$$

   where f and g are arbitrary functions.

4. *Forced wave equation.* An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\delta(x - x_0)$, where V is a constant. Using D'Alembert's solution, show that the position of the string for $t > 0$ is $y(x, t) = \frac{V}{2c}H(ct - |x - x_0|)$, where H is the Heaviside function. Now use the method of images to solve the same problem for a semi-infinite string $x \geq 0$ where the end at $x = 0$ is held fixed.

5. *Diffusion equation with a boundary source.* Consider the diffusion problem on the half-line:

$$\theta_t - D\theta_{xx} = f(x, t),\ 0 < x < \infty,\ 0 < t < \infty,$$

   with boundary and initial data $\theta(0, t) = h(t)$, $\theta(x, 0) = \Theta(x)$. This models the following scenario: suppose you have a metal rod at $x > 0$ with thermal diffusivity D. The initial temperature distribution is $\Theta(x)$ and you heat the rod with intensity $f(x, t)$. There is a time-dependent Dirichlet boundary condition that the temperature at the $x = 0$ end must be $h(t)$. Find the temperature of the rod $\theta(x, t)$.

   First make the change of variable $V(x, t) = \theta(x, t) - h(t)$ and transform the boundary condition and initial condition.

Then let $V(x,t) = V^0(x,t) + V^f(x,t)$ with appropriate boundary/initial conditions (use the lecture notes to help you choose suitable conditions when $t = 0$) and use the method of images.

The basic idea is to split the problem into a homogeneous problem with inhomogeneous initial condition and an inhomogeneous problem with homogeneous initial condition.

Show that

$$V_0(x,t) = \int_0^\infty [\Theta(u) - h(0)][K(x-u,t) - K(x+u,t)]\,du$$

$$V_f(x,t) = \int_0^t \int_0^\infty [K(x-\xi,t-\tau) - K(x+\xi,t-\tau)][f(\xi,\tau) - h'(\tau)]\,d\xi\,d\tau,$$

where $K(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right)$ is the heat kernel.

6. *Cauchy problem in the half-plane for the Laplacian.*

   (a) *Representation formula in 2D.* If u is a harmonic function in a 2D domain $\mathcal{D}$, with boundary $\partial\mathcal{D}$, show that

   $$u(\boldsymbol{\xi}) = \frac{1}{2\pi} \oint_{\partial\mathcal{D}} \left[u(\mathbf{x}) \frac{\partial}{\partial n}(\log|\mathbf{x} - \boldsymbol{\xi}|) - \log|\mathbf{x} - \boldsymbol{\xi}| \frac{\partial u}{\partial n} \right] dl,$$

   where dl is an arc element of $\partial\mathcal{D}$, $\mathbf{x} \in \partial\mathcal{D}$, $\boldsymbol{\xi} \in \mathcal{D}$.

   (b) If u is a harmonic function in a 2D domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$, and $G(\mathbf{x};\boldsymbol{\xi})$ is the Green's function for $\mathcal{D}$, show that

   $$u(\boldsymbol{\xi}) = \oint_{\partial\mathcal{D}} u(\mathbf{x}) \frac{\partial G}{\partial n}\,dl.$$

   (c) Consider Laplace's equation in the half-plane with prescribed boundary conditions at $y = 0$, i.e.

   $$\nabla^2\psi = 0;\ -\infty < x < \infty, y \geq 0,$$

   where $\psi(x,0) = f(x)$ a known function, such that ψ tends to zero as $y \to \infty$. Find the Green's function for this problem.

   (d) Hence show that the solution is given by another Poisson's integral formula:

   $$\psi(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2}\,d\xi.$$

   (e) Derive the same result by taking Fourier transforms with respect to x (assuming all transforms exist).

   (f) Find (in closed form) and sketch the solution for various $y > 0$ when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch the solution along $x = \pm a$.

   (g) Comment on what happens when you send $a \to \infty$.

7. *Heated Garden.* I stand in my garden, which may be viewed as the two-dimensional quarter plane $x, y > 0$, with heat source of strength Q positioned at the point (x_0, y_0). At $x = 0$ there is a conducting fence which is held at temperature T_0. At $y = 0$ there is an insulated house wall across which no heat can flow (i.e. the heat flux normal to the wall $-k\nabla T \cdot \mathbf{n}$ must vanish, where k is the diffusivity constant). The relationship between heat strength and heat flux is $Q = \int_{\partial S} -k\nabla T \cdot d\mathbf{l}$ (amount of heat produced within a region S is equal to

the heat flux through ∂S). Show that the equation and boundary conditions satisfied by the temperature field are

$$-\nabla^2 T = \frac{Q}{k}\delta(\mathbf{x}-\mathbf{x}_0), \qquad T = T_0 \text{ on } x=0, \ \frac{\partial T}{\partial y} = 0 \text{ on } y=0.$$

By using the method of images, find the temperature field.

Hence show that the magnitude of the heat flux across the fence at the point $(0, y)$ is

$$\frac{Qx_0}{\pi}\left[\frac{1}{x_0^2+(y-y_0)^2}+\frac{1}{x_0^2+(y+y_0)^2}\right].$$

Calculate the total heat radiated across the fence $x = 0$.

8. *Laplacian in cylindrical polar coordinates.* Consider the problem $\nabla^2 u = 0, r \neq 0, u \to 0$ as $r \to \infty$. Show that a solution of this equation which is independent of polar angle is $u_1 = 1/r = 1/(\rho^2+z^2)^{1/2}$ where ρ is the radial component in cylindrical polar coordinates. By considering the Laplacian in cylindrical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2},$$

and separating variables, show that, for an arbitrary function $f(\lambda)$,

$$u_2 = \int_0^\infty f(\lambda)e^{-\lambda|z|}J_0(\lambda\rho)\,d\lambda,$$

is also a solution which is independent of polar angle. By requiring $u_2 = u_1$, and then comparing these solutions on the axis $\rho = 0$, show that $f(\lambda) = 1$ is an admissible choice for $f(\lambda)$ and hence that

$$\int_0^\infty e^{-\lambda|z|}J_0(\lambda\rho)\,d\lambda = \frac{1}{\sqrt{\rho^2+z^2}}.$$

This is effectively a derivation of the **Laplace transform** of $J_0(\lambda\rho)$.

9. *Dirichlet Green's function for the sphere.*

   (a) Show that the Dirichlet Green's function for the Laplacian for the **interior** of a spherical domain of radius a is

   $$G(\mathbf{x};\mathbf{x}_0) = \frac{-1}{4\pi|\mathbf{x}-\mathbf{x}_0|}+\frac{a}{|\mathbf{x}_0|}\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0^*|}, \ \mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{|\mathbf{x}_0|^2}$$

   (b) Derive the Dirichlet Green's function for the Laplacian for the interior of the 2-dimensional disc $0 \le r \le a$;

   (c) and do the same for the half-disc $0 \le r \le a$, $0 \le \theta \le \pi$.

10. *Application of Green's function for a disk.* * Using Green's identity and the Dirichlet Green's function on the disk, show that the solution of

$$\nabla^2\Phi = 0 \text{ in } 0 \le r < a, \quad \Phi = \Psi(\theta) \text{ on } r = a,\ 0 \le 0 < 2\pi,$$

is

$$\Phi(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{\Psi(\theta')}{a^2-2ar\cos(\theta-\theta')+r^2}\,d\theta'.$$

Hence show that for $0 \le r < 1$,

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta'}{1 - 2r\cos(\theta - \theta') + r^2} &= \frac{2\pi}{1 - r^2}; \\
\int_0^{2\pi} \frac{\sin\theta'\, d\theta'}{1 - 2r\cos(\theta - \theta') + r^2} &= \frac{2\pi r \sin\theta}{1 - r^2}; \\
\int_0^{2\pi} \frac{\cos^2\theta'\, d\theta'}{1 - 2r\cos(\theta - \theta') + r^2} &= \frac{2\pi r^2 \cos^2\theta}{1 - r^2} + \pi.
\end{aligned}$$

Even if you don't want to do the derivation, the pretty application to these last 3 equations brings the course full circle, if you will excuse the pun.

Last edited: September 21, 2021