Methods Examples Sheet 3

Green's Function

1. *Boundary value problem.* Obtain the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \lambda^2 G = \delta(x - \xi), \quad 0 \leq x \leq 1, \quad 0 \leq \xi \leq 1,$$

where λ is real, subject to the boundary condition $G(0, \xi) = G(1, \xi) = 0$. *Make your life easier by choosing a convenient linear combination of complementary functions on the two intervals, i.e. instead of writing*

$$G(x;\xi) = \begin{cases} A\cosh\lambda x + B\sinh\lambda x & x \in [0, \xi] \\ C\cosh\lambda x + D\sinh\lambda x & x \in [\xi, 1], \end{cases}$$

use

$$G(x;\xi) = \begin{cases} A\cosh\lambda x + B\sinh\lambda x & x \in [0, \xi] \\ C\cosh(\lambda(1-x)) + D\sinh(\lambda(1-x)) & x \in [\xi, 1]. \end{cases}$$

Show that the solution to the equation

$$\frac{d^2y}{dx^2} - \lambda^2 y = f(x), \quad y(0) = y(1) = 0,$$

is

$$y(x) = -\frac{1}{\lambda\sinh\lambda}\left\{\sinh\lambda x\int_x^1 f(\xi)\sinh(\lambda(1-\xi))d\xi + \sinh(\lambda(1-x))\int_0^x f(\xi)\sinh\lambda\xi d\xi\right\}.$$

2. *Initial value problem.* Let us revisit the oil tanker problem which you have encountered in *IA Differential Equations*. The depth $y(t)$ of an oil tanker under an external force $f(t)$ satisfies the damped wave equation

$$\mathcal{L}y \coloneqq \ddot{y} + 2p\dot{y} + (p^2 + q^2)y = f(t),$$

where p, q are constants with $p > 0, q \neq 0$. Initially the oil tanker is at rest at the equilibrium so that $y(0) = \dot{y}(0) = 0$. In *IA Differential Equations* we solve this by finding the complementary functions (damped sinusoidal functions) and then guessing a particular integral. However guessing does not always work. We are going to solve for general f using two methods.

(a) The first method is to use the Green's function. The *Green's function* for this equation is defined to be the family of solutions $\{G(t;\tau)\}_{\tau>0}$ satisfying

$$\mathcal{L}G(t;\tau) = \delta(t-\tau), \qquad G(0;\tau) = \dot{G}(0;\tau) = 0,$$

i.e. the resulting depth if the oil tanker experiences an impulse at time $t = \tau$, modelled by $f(t) = \delta(t-\tau)$. Solve for the Green's function using methods from *IA Differential Equations*, and sketch it. Hence deduce that if the forcing term $f(t)$ is general, then the depth of the oil tanker is given by

$$y(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)}\sin[q(t-\tau)]f(\tau)d\tau.$$

(b) Now solve the same problem with the *Fourier transform* method. [*The Fourier transform method can be thought of decomposing any forcing term into the superposition of uncountably many mice exercising on the oil tanker, where each mouse is exerting a sinusoidal force of a different frequency and phase. *Also think of an animal metaphor for the decomposition of the forcing term as uncountably many Dirac deltas* $\delta(t-\tau)$.]

3. *Finite asymptotics.* A linear differential operator is defined by

$$L_x y = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y.$$

By writing $y = z/x$ or otherwise, find those solutions of $L_x y = 0$ which are either (a) bounded as $x \to 0$, or (b) bounded as $x \to \infty$. Find the Green's function $G(x,a)$ satisfying

$$L_x G(x,a) = \delta(x-a),$$

and both conditions (a) and (b). Use $G(x,a)$ to solve (subject to conditions (a) and (b)) $L_x y(x) = 1_{0\le x\le R}$, where 1_A denotes the indicator function of a set A. Show that the solution has the form, for suitable constants A, B

$$y(x) = \begin{cases} 1 + Ax^{-1}\sinh x, & \text{for } 0 \le x \le R, \\ Bx^{-1}e^{-x}, & \text{for } x > R. \end{cases}$$

4. *Higher order initial value problem.** Show that the Green's function for the initial value problem

$$\frac{d^4y}{dt^4} + k^2\frac{d^2y}{dt^2} = f(t), \quad y(0) = \dot{y}(0) = \ddot{y}(0) = y^{(3)}(0) = 0,$$

is given by

$$G(t,\tau) = \begin{cases} 0, & 0 \le t \le \tau \\ k^{-2}(t-\tau) - k^{-3}\sin k(t-\tau), & t \ge \tau. \end{cases}$$

Therefore, write down the integral form of the solution when $f(t) = e^{-t}$.

The Dirac delta function

5. *Delta function properties.* The continuously differentiable function $\phi(x)$ is monotone increasing in $[a,b]$ and has a simple zero at $x = c$ i.e. $\phi'(c) \neq 0$ where $a < c < b$. Show that

$$\int_a^b f(x)\delta(\phi(x))dx = \frac{f(c)}{|\phi'(c)|}.$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general (sufficiently nice) $\phi(x)$ provided the zeros are simple. Deduce that $\delta(at) = \delta(t)/|a|$ for $a \neq 0$. Also establish that

$$\int_{-\infty}^{\infty} |x|\delta(x^2 - a^2)dx = 1.$$

6. *Delta function derivative**. Show using polar coordinates that

$$\int_{\mathbb{R}^2} f(x^2+y^2)\delta'(x^2+y^2-1)\delta(x^2-y^2)\,dx\,dy = f(1) - f'(1).$$

Fourier transforms *Fourier transforms are useful, elegant and uncomplicated. They may be used in electrical problems or for mathematical interest. Inverse Fourier transforms in general need to be evaluated using contour integrals which are covered in the final section of Complex Methods.*

7. Calculate the Fourier transform of $f_1(x) = 1_{|x|\leq c}$. Hence, without integration, find the Fourier transforms of $f_2(x) = e^{iax}1_{|x|\leq c}$; $f_3(x) = \sin(ax)1_{|x|\leq c}$; $f_4(x) = \cos(ax)1_{|x|\leq c}$.

8. *Functions with discontinuities.* Let $f(x) = e^{-x}1_{x>0}$. Show that $\tilde{f}(k) = \frac{1-ik}{1+k^2}$. Show that the inverse Fourier transform of this Fourier transform $\tilde{f}(k)$ takes the value of $1/2$ at $x = 0$. (This is a general property of Fourier transforms, analogously to Fourier series. Inversion for general x is really straightforward with Complex Methods.)

9. *Fourier transform of Gaussians.* Find the Fourier transform of a Gaussian distribution $f(x) = e^{-n^2(x-\mu)^2}$, by differentiating both sides then taking the Fourier transform of both sides. *[You may quote the result for the Fourier transform of $xf(x)$.]*

   *Now attempt to calculate the Fourier transform using completing the square. Why is this not valid (without contour integration)?

   Now let $\mu = 0$, and consider $\delta_n(x) = \frac{n}{\sqrt{\pi}}f(x)$. Sketch $\delta_n(x)$ and $\tilde{\delta}_n(k)$ for small and large n. What is $\int_{\mathbb{R}} \delta_n(x)\,dx$? What is happening as $n \to \infty$?

10. *Parseval's relation continued.* By considering the Fourier transform of the function $f(x) = \cos(x)$ for $|x| < \pi/2$ and $f(x) = 0$ for $|x| \geq \pi/2$, and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\frac{\pi^2}{4}\cos^2 t}{(\frac{\pi^2}{4} - t^2)^2}\,dt = \int_0^\infty \frac{t^2\cos^2 t}{(\frac{\pi^2}{4} - t^2)^2}\,dt = \frac{\pi}{4}.$$

11. *Laplace's equation.* Show that the inverse Fourier transform of the function $\tilde{f}_\alpha(k) = (e^{k\alpha} - e^{-k\alpha})1_{|k|\leq 1}$ is

$$f_\alpha(x) = \frac{2i}{\pi(\alpha^2 + x^2)}(\alpha\cosh\alpha\sin x - x\cos x\sinh\alpha).$$

   Here α is a real constant. Determine, by using Fourier transforms, the solution of Laplace's equation in the infinite strip $0 \leq y \leq 1$, i.e. $\nabla^2\psi = 0$, $-\infty < x < \infty$, $0 < y < 1$, where $\psi(x, 0) = g(x) := \frac{2}{\pi(1+x^2)}(\cosh 1\sin x - x\cos x\sinh 1)$, and $\psi(x, 1) = 0$ for $-\infty < x < \infty$.

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