Methods Examples Sheet 1

Fourier Series *To remember the formula, it is best to remember the form when the interval is centred at 0, say $[-L, L]$, so that L is the radius of the interval. Then a $2L$-periodic function $f(x)$ can be decomposed as*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$ *and* $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$. *Most of the time we can take advantage of symmetry to argue that some of the coefficients are zero. For those that are not zero, use integration by parts (the quick tabular method, of course).*

1. *Fourier coefficients (full-range series).* For the periodic function $f(x) = (x^2 - 1)^2$ on the interval $-1 \leq x < 1$, show that it has Fourier series

$$f(x) = \frac{8}{15} + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos n\pi x.$$

Sketch the function $f(x)$ and comment on its differentiability and the order of the terms in its Fourier series as $n \to \infty$.

2. *Fourier coefficients (half-range series).* Suppose that $f(x) = x^2$ for $0 \leq x < \pi$. Express $f(x)$ as (a) a Fourier sine series, and (b) a cosine series, each having period 2π. Sketch the functions represented by (a) and (b) in the range -6π to 6π. If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for $g(x) = 2x$ and $h(x) = 2|x|$ each in the range $[-\pi, \pi)$?

3. *Series summation.* Find the Fourier series of the 2π-periodic function $f(x) = e^x$ on $[-\pi, \pi)$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{2}(\pi \coth \pi - 1).$$

Neither a_n nor b_n vanish by symmetry, so it is best to use the complex version of Fourier series to find both at the same time. Recall $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$ where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx.$$

The coefficients a_n, b_n can be obtained by taking real and imaginary parts.

4. *Parseval's identity and a low pass filter.* (i) Given that a function $f(t)$ defined over the interval $(-T, T)$ has the Fourier series

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right],$$

show that

$$\frac{1}{T} \int_{-T}^{T} (f(t))^2 \, dt = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

where you may assume $f(t)$ is such that this series is convergent. This quantity is proportional to the energy.

(ii) A unit amplitude square wave of period $2T$ is given by $f(t) = 1$ for $0 < t < T$ and $f(t) = -1$ for $-T < t < 0$. Suppose this is the input for a system which permits angular frequencies less than $4.5\pi T^{-1}$ to be perfectly transmitted and frequencies greater than $4.5\pi T^{-1}$ to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of $f^2(t)$; what fraction of the incident power is transmitted?

5. *Discontinuities and the Wilbraham-Gibbs phenomenon.**

   (a) Suppose that f is a 2π-periodic square wave given by

   $$f(x) = \begin{cases} 1 & x \in (0, \pi) \\ 0 & x \in (\pi, 2\pi). \end{cases}$$

   Sketch f and show that its Fourier series is

   $$\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

   (b) Now define the partial sum of this series as

   $$S_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1},$$

   and prove, by considering $\sum_{n=1}^{N} \cos(2n-1)x$, the following:

   $$S_N(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^x \frac{\sin 2Nt}{\sin t}\, dt.$$

   (c) Deduce that $S_N(x)$ has extrema at $x = m\pi/2N$, $m = 1, 2, \ldots, 2N-1, 2N+1, \ldots$, i.e. all integer m except multiples of $2N$, and that the height of the first maximum for large N is approximately

   $$S_N\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin u}{u}\, du \approx 1.089.$$

   Comment on the accuracy of Fourier series at discontinuities. (This question takes you through some important steps which are used in the proof of Fourier's theorem – refer, for example, to chapter 14 of Jeffreys and Jeffreys. Henry Wilbraham was a fellow of Trinity who wrote a paper when he was 22, deriving this result 50 years before Gibbs, after whom it is commonly named.)

Sturm-Liouville Theory

6. *Eigenfunctions and eigenvalues.* Prove that the Robin boundary value problem for the heat operator

   $$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) + y'(1) = 0,$$

   has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3, \ldots$. Indicate roughly the behaviour of λ_n as $n \to \infty$. Hence write down the solution to $-y'' = f$, subject to $y(0) = 0$ and $y(1) + y'(1) = 0$.

7. *Recasting in SL form.*

   (a) $(1 - x^2)y'' - 2xy' + \lambda y = 0$;

   (b) $x(x-1)y'' + (ax - b)y' + \lambda y = 0$, where $a > b$;

   (c) Find the eigenvalues and eigenfunctions for $y'' + 4y' + (4 + \lambda)y = 0$ with $y(0) = y(1) = 0$. What is the orthogonality relation for these eigenfunctions?

8. *Bessel's equation.* This comes up in the the diffusion equation/wave equation in plane polars. See sheet 2.

(a) *Show that one solution to $\frac{d}{dz}(z\frac{dy}{dz}) + zy = 0$ for $z > 0$ is

$$J_0(z) := \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m}(m!)^2},$$

and the second solution is a sum of a regular function and $J_0(z)\log z$.

(b) Define $(\mathcal{L}u)(x) = \frac{1}{x}(-xu'(x))'$, with $u(x)$ bounded as $x \to 0$ and $u(1) = 0$. Show that $\mathcal{L}$ is self-adjoint with respect to the inner product with weight function $w(x) = x$:

$$(f, g)_w := \int_0^1 xf(x)g(x)dx.$$

(c) Show that the eigenvalues of the Sturm-Liouville problem $\mathcal{L}u = \lambda u$, in the domain $0 < x < 1$ are $\lambda = j_n^2$ for $n = 1, 2, \ldots$, where j_n are the zeros of the Bessel function $J_0(z)$, arranged in ascending order.

(d) Using integration by parts on the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$, verify that the eigenfunctions are orthogonal with respect to the inner product defined above. Find also the norm of the eigenfunctions $J_0(j_n x)$ corresponding to the inner product defined above.

Hint: show

$$\int_0^1 xJ_0(\alpha x)J_0(\beta x)dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2} \quad \beta \neq \alpha,$$

$$\int_0^1 xJ_0(j_n x)J_0(j_m x)dx = 0, \quad n \neq m,$$

$$\int_0^1 xJ_0(j_n x)^2 dx = \frac{1}{2}J_0'(j_n)^2.$$

For the last one, consider $\beta = j_n + \epsilon$ as $\epsilon \to 0$.

(e) Assume that the inhomogeneous equation $\mathcal{L}u - \tilde{\lambda}u = f$, where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution such that u is bounded as $x \to 0$ and $u(1) = 0$. Assuming also that $f(x)$ satisfies the same boundary conditions as u and the completeness of the eigenfunctions $J_0(j_n x)$, obtain the eigenfunction expansion of u.

9. *Higher order self-adjoint form.* Show that the fourth-order differential operator

$$\mathcal{L} = \sum_{r=0}^{4} p_r(x)\frac{d^r}{dx^r},$$

where $p_r(x)$ are real functions, is self-adjoint (for appropriate choices of boundary conditions, which you do not need to determine) only if $p_3 = 2p_4', p_1 = p_2' - p_4'''$.

Consider a specific example, show that the boundary value problem

$$-y'''' + \lambda y = 0; \quad y(0) = y(1) = y'(0) = y'(1) = 0,$$

where $\lambda > 0$ is real, has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$. Indicate roughly the behaviour of λ_n as $n \to \infty$.

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