Mathematical Tripos Part IB H.S. Reall

ELECTROMAGNETISM Examples 3 Lent 2022

1. A monochromatic plane wave, propagates in empty space $z < 0$ with fields

$$\mathbf{E}_{\text{inc}} = \mathbf{e}_x \text{Re}\left(\alpha e^{i(kz-\omega t)}\right) \qquad \mathbf{B}_{\text{inc}} = \frac{1}{c}\mathbf{e}_y \text{Re}\left(\alpha e^{i(kz-\omega t)}\right)$$

A perfect conductor fills the region $z \geq 0$. Show that if the reflected fields are given by

$$\mathbf{E}_{\text{ref}} = -\mathbf{e}_x \text{Re}\left(\alpha e^{i(-kz-\omega t)}\right) \qquad \mathbf{B}_{\text{ref}} = \frac{1}{c}\mathbf{e}_y \text{Re}\left(\alpha e^{i(-kz-\omega t)}\right)$$

then the total fields $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$ and $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{ref}}$ satisfy the Maxwell equations and the relevant boundary conditions at $z = 0$.

What surface current flows in the plane $z = 0$? Compute the Poynting vector in the region $z < 0$ and determine its value averaged over a period $T = 2\pi/\omega$.

Recall from Q5 of sheet 2 that a surface current experiences a Lorentz force from the average magnetic field on either side of the surface. Use this to show that the time-averaged force per unit area on the conductor is $\langle f \rangle = \epsilon_0 |\alpha|^2$.

2. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic plane wave can propagate between the plates in the y direction only if the frequency is given by $\omega = n\pi c/a$ with $n \in \mathbf{Z}$.

3. Perfectly conducting plates are positioned at $y = 0$ and $y = a$. Show that a monochromatic wave may propagate between the plates in the direction z if the field components are

$$E_x = \omega A \sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t)$$

and

$$B_y = kA\sin\left(\frac{n\pi y}{a}\right)\sin(kz - \omega t) \qquad B_z = \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\cos(kz - \omega t)$$

with A a constant and $n \in \mathbf{Z}$. Show that the wavelength λ is given by $1/\lambda^2 = 1/\lambda_\infty^2 - n^2/4a^2$, where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plates.

4. Consider a plane polarized electromagnetic wave described by the vector and scalar potentials $\mathbf{A}(t, \mathbf{x}) = \text{Re}\left(\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ and $\Phi(t, \mathbf{x}) = \text{Re}\left(\Phi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$ with constant $\mathbf{A}_0$ and Φ_0. Use Maxwell's equations to find a relationship between $\mathbf{A}_0$ and Φ_0.

Find a gauge transformation such that the new vector potential is "transversely polarised", i.e. $\mathbf{A}_0 \cdot \mathbf{k} = 0$. What is the scalar potential Φ in this gauge?

5. (a) A tensor of type $(0, 2)$ has components $T_{\mu\nu}$. View these components as a 4×4 matrix. Show that if this matrix is invertible in one inertial frame then it is invertible in any inertial frame, and that the components of the inverse matrix $(T^{-1})^{\mu\nu}$ define a tensor of type $(2, 0)$.

(b) Show that the object with components $\epsilon_{\mu\nu\rho\sigma}$ w.r.t. any inertial frame is an isotropic pseudo-tensor of type $(0, 4)$.

6. A particle of rest mass m and charge q moves in a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. It starts from the origin with initial 3-momentum $\mathbf{p} = (0, p_0, 0)$. Show that the particle traces out a path in the (x, y) plane given by

$$x = \frac{\mathcal{E}_0}{qE}\left(\cosh\left(\frac{qEy}{p_0 c}\right) - 1\right)$$

where $\mathcal{E}_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ is the initial kinematic energy of the particle.

7. For constant electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, show that if $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{E}^2 - c^2\mathbf{B}^2 \neq 0$ then there exist inertial frames where either $\mathbf{E}$ or $\mathbf{B}$ are zero, but not both. [Hint: show that you can choose axes so that only E_y and B_z are non zero and then consider a Lorentz transformation in the x-direction.]

8. An electromagnetic wave is reflected by a perfect conductor at $x = 0$. The electric field is $\mathbf{E}(t, \mathbf{x}) = \mathbf{e}_y\left[f(t_-) - f(t_+)\right]$ where f is an arbitrary function and $ct_\pm = ct \pm x$. Show that this satisfies the relevant boundary condition at the conductor. Find the corresponding magnetic field $\mathbf{B}$.

   Show that under a Lorentz transformation to an inertial frame moving with speed v in the x-direction the electric field is transformed to

$$\mathbf{E}'(t', \mathbf{x}') = \mathbf{e}_y\left[\rho f(\rho t'_-) - \frac{1}{\rho} f\left(\frac{t'_+}{\rho}\right)\right] \quad \text{where} \quad \rho = \sqrt{\frac{c-v}{c+v}}$$

   Hence for an incident wave $\mathbf{E}(t, \mathbf{x}) = \mathbf{e}_y F(t_-)$, find the wave that is reflected after it hits a perfectly conducting mirror moving with speed v in the x-direction.

9. (a) A scalar field Φ obeys the wave equation $\partial^\mu \partial_\mu \Phi = 0$. Its *energy-momentum tensor* is $T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi$. Show that $T_{\mu\nu}$ is *conserved*: $\partial_\nu T^{\mu\nu} = 0$.

   (b) The energy-momentum tensor of the Maxwell field is $T_{\mu\nu} = \mu_0^{-1}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$. Explain how T_{00} and T_{0i} are related to the energy density and Poynting vector of the electromagnetic field. Show that Maxwell's equations imply that $\partial_\nu T^{\mu\nu} = -F^\mu{}_\nu j^\nu$ and that the time component of this equation is the energy conservation equation for Maxwell's theory.

10. ($\star$) For a general 4-velocity, written as $U^\mu = \gamma(c, \mathbf{v})$, show that

$$F^{\mu\nu}U_\nu = \gamma\begin{pmatrix} \mathbf{E}\cdot\mathbf{v}/c \\ \mathbf{E} + \mathbf{v}\times\mathbf{B} \end{pmatrix}$$

   In the rest-frame of a conducting medium, Ohm's law states that $\mathbf{J} = \sigma\mathbf{E}$ where σ is the conductivity and $\mathbf{J}$ is the 3-current. Assuming that σ is a Lorentz scalar, show that Ohm's law can be written covariantly as

$$j^\mu + \frac{1}{c^2}(j^\nu U_\nu)U^\mu = \sigma F^{\mu\nu}U_\nu$$

   where j^μ is the charge-current density and U^μ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho\mathbf{v} + \sigma\gamma\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{c^2}(\mathbf{v}\cdot\mathbf{E})\mathbf{v}\right)$$

   where ρ is the charge density. Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.