MATHEMATICAL TRIPOS: PART IA **Lent 2022**

PROBABILITY **Dominic Yeo**

Example Sheet 2 (of 4)

1. Let $A, B \in \mathcal{F}$ be two events such that $\mathbb{P}(B) = 0$ or 1. Show that A and B are independent.

[*Note, this generalises the result in lectures where $B = \emptyset$ or $B = \Omega$.*]

2. A coin with probability $p \in [0,1]$ of heads is tossed n times. Let E be the event 'a head is obtained on the first toss' and F_k the event 'exactly k heads are obtained'. For which pairs of non-negative integers (n, k) are E and F_k independent?

3. Independent trials are performed, each with probability p of success. Let P_n be the probability that n trials result in an even number of successes. Show that

$$P_n = \tfrac{1}{2}(1 + (1-2p)^n).$$

4. Two players A and B throw darts at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws A has probability p_A and B has probability p_B of scoring a bull.

(a) Suppose they play separately, and A first scores a bull on throw X_A, while B first succeeds on throw X_B. Find the distribution of $Z = \min(X_A, X_B)$.

(b) Now suppose they throw alternately, and the first to score a bull wins the contest. If A has first throw, calculate the probability p that A wins.

5. Consider the probability space $\Omega = \{0,1\}^3$ with equally likely outcomes.

(a) Show that there are 70 different Bernoulli random variables of parameter $1/2$ that can be defined on Ω.

(b) How many Bernoulli random variables of parameter $1/3$ can be defined on Ω?

(c) What is the length of the longest sequence of independent Bernoulli random variables of parameter $1/2$ that can be defined on Ω?

6. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find the distribution of $X + Y$. Prove that the conditional distribution of X, given that $X + Y = n$, is binomial with parameters n and $\lambda/(\lambda + \mu)$.

7. The number of misprints on each page has a Poisson distribution with parameter λ, and the numbers on different pages are independent.

(a) What is the probability that the second misprint will occur on page r?

(b) A proof-reader studies a single page looking for misprints. She catches each misprint (independently of others) with probability $p \in [0,1]$. Let X be the number of misprints she catches and let Y be the number she misses. Find the distributions of the random variables X and Y and show they are independent. Compare with the previous exercise.

8. Let $X_1, \dots, X_n$ be independent identically distributed random variables with mean μ and variance σ^2. Find the means of the random variables

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2,$$

and the variance of $\bar{X}$.

9. Sarah collects figures from cornflakes packets. Each packet contains one of n distinct figures. Each type of figure is equally likely. Show that the expected number of packets Sarah needs to buy to collect a complete set of n is

$$n\sum_{i=1}^{n}\frac{1}{i}.$$

10. Let $a_1, a_2, \dots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years. Assume that $a_1, a_2, \dots, a_n$ is a random permutation of $1, 2, \dots, n$. Say that k is a record year if $a_k < a_i$ for all $i < k$. Thus the first year is always a record year. Let $Y_i = 1$ if i is a record year and 0 otherwise. Find the distribution of Y_i and show that $Y_1, Y_2, \dots, Y_n$ are independent. Calculate the mean and variance of the number N of record years in the next n years.

11. A fair coin is tossed $n+1$ times. For $1 \le i \le n$, let A_i be the event that the ith and $(i+1)$th outcomes are both heads.

(a) Find $\mathbb{P}(A_i \cap A_j)$ for all $1 \le i \ne j \le n$.

(b) Define $M = \mathbf{1}_{A_1} + \dots + \mathbf{1}_{A_n}$, the number of occurrences of HH in the sequence. Find the mean and variance of M. (Hint: you need to use covariances!)

(c) Similarly, find the mean and variance of the number of occurrences of TH in the sequence.

12. Let $s \in (1, \infty)$ and let X be a random variable in $\{1, 2, \dots\}$ with distribution

$$\mathbb{P}(X = n) = n^{-s}/\zeta(s)$$

where $\zeta(s)$ is a suitable normalizing constant. For each prime number p let A_p be the event that X is divisible by p. Find $\mathbb{P}(A_p)$ and show that the events $(A_p : p \text{ prime})$ are independent. Deduce that

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$

Extensions

13. Let X be a geometric random variable on $\{0, 1, 2, \ldots\}$ with parameter $p \in (0, 1)$, and Y a Poisson random variable with parameter $\lambda > 0$. Compare the distributions of i) $X - n$, conditional on $X \geq n$; and ii) $Y - n$, conditional on $Y \geq n$, when n is large.

14. Can you construct non-negative integer valued random variables X and $(X_n)_{n\geq 1}$ with $\mathbb{E}[X], \mathbb{E}[X_n]$ all finite, and such that

$$\mathbb{P}(X_n = k) \; \to \; \mathbb{P}(X = k) \text{ as } n \to \infty,$$

holds for every $k \in \{0, 1, \ldots\}$, but for which $\mathbb{E}[X_n] \not\to \mathbb{E}[X]$?

15. Let σ_n be a uniformly chosen permutation from Σ_n, and consider the cycle decomposition of σ_n. Let $\alpha_{n,k}$ be the number of cycles in σ_n of length k. Find $\mathbb{E}[\alpha_{n,k}]$. Let α_n be the total number of cycles in σ_n. Show that $\mathbb{E}[\alpha_n] \to \infty$ as $n \to \infty$.

Now, let ℓ_n be the length of the cycle of σ_n which includes the element 1. Find the distribution of ℓ_n, and also $\mathbb{E}[\ell_n]$.

Note that $\mathbb{E}[\ell_n]\mathbb{E}[\alpha_n] \gg n$. Explain why this is not a contradiction.

16. In a community of N people, birthdays are independent, and uniformly chosen from the 365 days of the non-leap year. How would you try to find an expression for the probability that at least k people share a birthday? (Ie, at least one day d such that at least k people were born on day d.)

Would this analysis be easier if instead you assumed the number of people in the community was random, with Poisson(N) distribution? (Hint: Exercise 7 is useful.)

What about if the birthdays were independent but not distributed uniformly?