Examples Sheet 4

You will need to be able to quickly find series solutions. They will reappear in IB Methods and Quantum Mechanics. In general you do not need to write the series solution with summation notation. It suffices to write down the first few terms, enough to show the pattern. By now you should be familiar with the series of well-known elementary functions, so simplify the series whenever possible.

1. Find two independent series solutions about $x = 0$ of

$$4xy'' + 2(1-x)y' - y = 0.$$

2. Find the two independent series solutions about $x = 0$ of

$$y'' - 2xy' + \lambda y = 0,$$

for a constant λ. Show that for $\lambda = 2n$, for n a positive integer, one of the solutions is a polynomial of degree n. These are the Hermite polynomials relevant for the solution of the simple harmonic oscillator in quantum mechanics.

3. Find a series solution about $x = 0$ for $\gamma \neq 0$ of

$$x^2y'' - xy' + (1-\gamma x)y = 0$$

and write down the form of a second, independent solution. Find two independent solutions of the equation when $\gamma = 0$.

4. Bessel's equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

For $\nu = 0$, find a solution in the form of a power series about $x = 0$.

For $\nu = \frac{1}{2}$, find two independent series solutions of this form. Perform also the change of variables $y(x) = z(x)/\sqrt{x}$ to simplify the equation, and solve for $z(x)$.

The Hessian tells you locally how convex or concave the stationary point is. If we have a few stationary points and their Hessian, we can infer approximately how the function might look like. It is not always necessary to compute the Hessian to determine the nature of a stationary point $\mathbf{x}_0 \in \mathbb{R}^n$. For example, if $f(\mathbf{x}_0) = 0$ and f is a non-negative function, then $\mathbf{x}_0$ can only be a minimum (without needing to compute the Hessian). As another example, if the stationary point is found at the intersection of two contour lines, then it must be a saddle point.

5. Find the position and nature of each of the stationary points of

$$f(x, y) = x^3 + 3xy^2 - 3x$$

and draw a rough sketch of the contours of f. *Hint: looks like ET.*

6. (a) Find the positions of each of the stationary points of $f(x, y) = \sin x \sin y$ in $(0, 2\pi) \times (0, 2\pi)$. By using this information and identifying the zero contours of f, sketch the contours of f and identify the nature of the stationary points.

   (b) By using a linear transformation, sketch $g(u, v) = \sin(\frac{u-v}{2}) \sin v$ in $(0, 2\pi) \times (0, 2\pi)$.

7. (a) Let $f(x, y) = x^2 + y^3 - 3y$. Find and classify the stationary points of f by finding the eigenvalues of the Hessian matrix of f.

   (b) What do the eigenvalues and eigenvectors tell you about the graph?

   (c) Sketch the contours of f and add to the sketch a few arrows showing the directions of ∇f.

   (d) Find the polynomial in x, y of degree ≤ 2 that best approximates f at each of the stationary points of f.

   (e) Sketch the contour surfaces of $g(x, y, z) = x^2 + y^2 + z^3 - 3z$ and write down the Hessian matrix at each stationary point.

We finally move onto solving ODEs in higher dimensions.

8. Let $x(t)$ and $y(t)$ be the amount of cell X and cell Y in the laboratory, respectively. Initially there are x_0 and y_0 grams of cell X and Y, respectively. Cell X decays at rate 1, and each cell Y creates a new cell X and a new cell Y at rate 1. The amounts $x(t)$ and $y(t)$ can be modelled as

$$\dot{x} = -x + y, \quad \dot{y} = y.$$

   (a) Find the amount of cell X and Y at time t (i.e. solve for $x(t)$ and $y(t)$ by writing the system as $\dot{\mathbf{x}} = M\mathbf{x}$).

   (b) Sketch the phase portrait (if you include negative x, y then it may look more familiar). It should be a saddle. Draw the solution trajectories and show their directions.

   (c) What ratio $x_0 : y_0$ should I start off with so that the ratio $x(t) : y(t)$ remains constant at all times?

   (d) For which initial conditions (x_0, y_0) will $x(t)$ decrease then increase?

The eigenvalues determine the nature of the fixed point (in the above case, the only fixed point is the origin), and the eigenvectors determine the direction of growth/decay, as you have shown above. Let us now explore the phase portrait of different types of matrices.

In systems involving more complicated dynamics, the ODE is usually not linear. For example, in the predator-prey model, quadratic terms are present, modelling for interaction terms. In this case, explicit solutions $x(t), y(t)$ are not readily available, but the phase portrait still provides us with a lot of information.

9. Carnivorous hunters of population y prey on vegetarians of population x. In the absence of hunters the prey will increase in number until their population is limited by the

availability of food. In the absence of prey the hunters will eventually die out. The equations governing the evolution of the populations are

$$\dot{x} = x - x^2 - xy, \qquad \dot{y} = \frac{y}{8}\left(\frac{x}{b} - 1\right),$$

where b is a positive constant, and $x(t)$ and $y(t)$ are non-negative functions of time t.

(a) Explain the origin of each term. What property of the carnivorous hunters does b control?

(b) In the two cases (i) $0 < b < 1/2$ and (ii) $b > 1$ determine the location and the stability properties of the critical points. In both of these cases sketch the typical solution trajectories and briefly describe the ultimate fate of hunters and prey.

10. The evolution of an infectious disease in a population can be modelled by

$$\dot{U} = U - U(U + I) - \beta UI$$
$$\dot{I} = I - I(U + I) + \beta UI - \delta I$$

where U is the uninfected population, I is the infected population, $\beta > 0$ is the rate of infection, and $\delta > 0$ is the death rate caused by the disease. For this problem set $\beta = \frac{3}{4}$.

(a) Explain the origin of each term.

(b) What is unrealistic for this model? *Open-ended question.*

(c) Determine the location and stability of the critical points of the above systems in the cases (i)$\delta = \frac{1}{5}$, (ii) $\delta = \frac{2}{5}$, (iii) $\delta = \frac{3}{5}$.

(d) Thus determine the long-term outcome for the population in each case.

(e) Which of these values of δ gives the least total population in the long term? Explain why this occurs. Which do you think is worse: a disease with a mortality rate of $\frac{2}{5}$ or of $\frac{3}{5}$?

11. The *convection-diffusion equation* is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial x} + R(x, t),$$

where $u(x, t)$ is the concentration of chemical/plankton/variable of interest in a fluid at position x and time t, D is the diffusivity, a is the velocity of the fluid, and $R(x, t)$ describes the source or sink of the quantity u. $R > 0$ means more species is created, $R < 0$ means species is being destroyed.

(a) Solve the linear transport equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},$$

for $u(x, t)$ given $u(x, 0) = g(x)$, where $a \in \mathbb{R}$ represents velocity. *Hint: find the contour lines of u, which are always perpendicular to the gradient if you visualise u in the x-t plane.* Draw the solution on a computer (e.g. desmos) for a $g(x)$ of your choice.

(b) Now solve

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + u - u^2,$$

for $u(x,t)$ given $u(x,0) = g(x)$. This could be modelling the density $u(x,t)$ of plankton at position x and time t which is flowing down a river at velocity a. Draw the solution on a computer (e.g. desmos) for a $g(x)$ of your choice.

(c) The diffusion equation is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

subject to $u(x,0) = g(x)$. Define the *fundamental solution*

$$F_t(x) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/4Dt},$$

which is a bell-shaped curve with variance $2Dt$. Verify that

$$u(x,t) = (g * F_t)(x) \coloneqq \int_{-\infty}^{\infty} g(y)F_t(x-y)\,dy$$

is a solution to the diffusion equation (in fact, the solution, because the solution is unique). The notation $g * F_t$ is called the *convolution* of g with F_t. Verify that $g * \delta = g = \delta * g$, where δ is the Dirac delta function. Can $*$ be made into a group operation?

(d) Develop a more realistic model for plankton growth in the ocean that includes the diffusion term. You can then consider many different ways of extensions. For example, you can include zooplankton and phytoplankton as two different species, so that we have a coupled PDE for the predator-prey model. You need not solve your PDE.

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