Examples Sheet 3

PART A

1. Find the general solutions of
   (a) $y'' + 5y' + 6y = e^{3x}$
   (b) $y'' + 9y = \cos 3x$
   (c) $y'' - 2y' + y = (x-1)e^x$.

2. The function $y(x)$ satisfies the linear equation $y''(x) + p(x)y'(x) + q(x)y(x) = 0$. The Wronskian $W(x)$ of two independent solutions, denoted $y_1(x)$ and $y_2(x)$, is defined to be $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Let $y_1(x)$ be given. By treating the definition of the Wronskian as a first-order inhomogeneous differential equation for $y_2(x)$, show that
$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{W(t)}{y_1(t)^2} dt. \qquad (*)$$
   Show that $W'(x) + p(x)W(x) = 0$.
   Verify that $y_1(x) = 1 - x$ is a solution of $xy'' - (1 - x^2)y' - (1 + x)y = 0$. Hence using $(*)$ with $x_0 = 0$ and expanding the integrand in powers of t to order t^3, find the first three nonzero terms in the power series expansion for a solution, y_2, that is independent of y_1 and satisfies $y_2(0) = 0, y_2''(0) = 1$.
   *What would happen if I chose a different value for x_0?

3. Given the solution $y_1(x)$, find a second solution of the following equations:
   (a) $x(x + 1)y'' + (x - 1)y' - y = 0$, $y_1(x) = (x + 1)^{-1}$;
   (b) $xy'' - y' - 4x^3y = 0$, $y_1(x) = e^{x^2}$.

4. Find the general solutions of
   (a) $y_{n+2} + y_{n+1} - 6y_n = n^2$,
   (b) $y_{n+2} - 3y_{n+1} + 2y_n = n$,
   (c) $y_{n+2} - 4y_{n+1} + 4y_n = a^n$, where $a \neq 2$. By expressing $a^n = (2 + (a - 2))^n$ as a Taylor series about $a = 2$ (or expanding using the binomial theorem), find the general solution the case $a = 2$.

5. Find the solution of $y'' - y' - 2y = 0$ that satisfies $y(0) = 1$ and is bounded as $x \to \infty$.
   Solve the related difference equation
$$(y_{n+1} - 2y_n + y_{n-1}) - \frac{1}{2}h(y_{n+1} - y_{n-1}) - 2h^2 y_n = 0,$$
   and show that if $0 < h \ll 1$ the solution that satisfies $y_0 = 1$ and for which y_n is bounded as $n \to \infty$ is approximately $y_n = (1 - h + \frac{1}{2}h^2)^n$. Explain the relation with the solution in the first part. *Hint: binomial expand* $(h^2 + \frac{9}{4})^{1/2}$.

6. Show that
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) \equiv \frac{1}{r}\frac{d^2}{dr^2}(rT)$$
   and hence solve the equation
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = k^2 T$$
   for $r \neq 0$ subject to the condition that $\lim_{r\to 0} T(r)$ is finite and $T(1) = 1$. Present your answer using hyperbolic functions.

7. *Variation of parameters.* Let $y(x)$ satisfy $y'' - 2x^{-1}y' + 2x^{-2}y = f(x)$ for some forcing term f. First find two independent solutions y_1, y_2 of the homogeneous equation (i.e. when $f = 0$). Find the most general solution in the case $f(x) = x \sin x$.

PART B

This section has longer questions.

8. A large oil tanker of mass M floats on the sea of density ρ. It is at rest at its equilibrium depth at time $t < 0$. Whenever the tanker is given a small downward displacement z ($z > 0$ when tanker is underwater), the upward force is equal to the weight of water displaced (Archimedes' Principle). The cross-sectional area A of the tanker at the water surface is constant.

   (a) Sketch a diagram. Show that this upward force is $g\rho Az$, and hence that

   $$\ddot{z} + \frac{g\rho A}{M} z = 0.$$

   Thus Archimedes' Principle implies that a floating object undergoes simple harmonic motion.

   (b) A wild mouse appears at $t = 0$ (which you should add to your diagram) and exercises on the deck of the tanker producing a vertical force $F_0 \cos \omega_0 t$, where $\omega_0 = (g\rho A/M)^{1/2}$. Solve for $z(t)$ and sketch it. Show that the oil tanker will eventually sink. *Guess a particular solution by inspection.*

   (c) In practice, as the vertical motion of the tanker increases, waves will be generated. This can be modelled by an additional damping force $2M\gamma\dot{z}$, where $\gamma \geq 0$ is the damping parameter. Before solving for $z(t)$, make a guess as to how the damping parameter might affect the solution. Will the value of γ determine whether the tanker sinks/does not sink? Write down the differential equation in this case. Now solve for $z(t)$ and sketch the solution. *Also guess a particular solution by inspection.*

   (d) *(Optional.)* Suppose instead the mouse produces a vertical force $F_1 \cos \omega_1 t$ where $\omega_1 \neq \omega_0$. What will happen now in the long term, both in the damped and undamped case?

   Hint: The differential equation modelling this is

   $$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \frac{F_1}{M} \cos \omega_1 t, \qquad z(0) = \dot{z}(0-) = 0.$$

   We guess the particular solution $z(t) = \alpha_1 \cos(\omega_1 t - \theta_1)$ and solve for α_1, θ_1. [This is equivalent to guessing $z(t) = A \cos \omega_1 t + B \sin \omega_1 t$, but the former is better for physical interpretation.] Substituting this into the differential equation and using trigonometric identities gives you

   $$\alpha_1 = \frac{F_1/M}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}, \qquad \theta_1 = \arctan \frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}.$$

   An easier method is to use complex numbers here. Consider a different differential equation:

   $$\ddot{w} + 2\gamma\dot{w} + \omega_0^2 w = \frac{F_1}{M} e^{i\omega_1 t}, \qquad w(0) = \dot{w}(0-) = 0.$$

   We can solve this by guessing $w(t) = \alpha_1 e^{i(\omega_1 t - \theta_1)}$, for $\alpha_1 \geq 0, \theta_1 \in [0, 2\pi)$. By comparing real and imaginary parts we can easily get the values of α_1 and θ_1 as above. Then the real part of $w(t)$ will satisfy the DE for $z(t)$. Finally, the particular solution is then combined with the complementary function (3 cases), whose coefficients are determined by the initial conditions, which you do not need to determine.

9. We investigate what happens to an object undergoing simple harmonic motion if we apply discontinuous external forces to it. Imagine your favourite simple harmonic oscillator (SHO), e.g. a spring or a simple pendulum, initially at rest at equilibrium. The distance from the equilibrium $y(t)$ satisfies

$$\ddot{y} + \omega^2 y = f(t), \qquad y(0) = 0 = \dot{y}(0) = 0,$$

where $f(t)$ is the external force and ω is the natural oscillation frequency.

(a) Suppose instead we hit it hard at time $t = a$ with force F_0. This can be modelled by $f(t) = F_0\delta(t-a)$. Is the velocity continuous at time $t = a$? Solve for y in this case and sketch it.

(b) Suppose we pull it with a constant force F_0 between time $t = a$ and $t = b$ $(a < b)$. This is modelled by $f(t) = F_0(H(t-b) - H(t-a))$. Solve for y in this case. Is the velocity continuous at times $t = a, b$? Sketch the case when $\omega, F_0 = 1$, $a = \pi$, $b = 2\pi$.

(c) Solve $\ddot{y} + 2\dot{y} + 5y = F_0\delta(t)$ given that $y = 0$ for $t < 0$. Sketch the solution. Give an example of a physical system that this describes (*Hint: look at the oil tanker and the SHO question*).

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