Examples Sheet 2

1. Show that the general solution of

$$y' - y = e^{ux}, \quad u \neq 1 \qquad (*)$$

can be written (by means of a suitable choice of A) in the form

$$y(x) = Ae^x + \frac{e^{ux} - e^x}{u - 1}.$$

By taking the limit as $u \to 1$ and using l'Hôpital's rule, find the general solution of $(*)$ when $u = 1$. What does this illustrate?

2. Solve the following.
   (a) $y'x \sin x + (\sin x + x \cos x)y = xe^x$;
   (b) $y' \tan x + y = 1$;
   (c) $y' = x^2(1 + y^2)$;
   (d) $y' = \cos^2 x \cos^2(2y)$;

3. Solve the following.
   (a) $y' = (x - y)^2$;
   (b) $(e^y + x)y' + (e^x + y) = 0$.
   (c) $y' = (e^y - x)^{-1}$;

4. Find all solutions of the equation $yy' - x = 0$ and give a sketch showing the solutions. By means of the substitution $y = \log u - x$, deduce the general solution of

$$(\log u - x)\frac{du}{dx} - u \log u = 0.$$

Sketch the solutions, starting from your previous sketch and drawing first the lines to which $y = \pm x$ are mapped.

5. In each of the following sketch a few isoclines.
   (a) $y' = x^2 + y^2$;
   (b) $y' = (1 - y)(2 - y)$;
   (c) $\frac{dy}{dx} = xy$. For this differential equation, also find the family of solutions determined by this equation and reassure yourself that your sketches were appropriate.

6. Sketch the isoclines for the equation

$$\frac{dy}{dx} = \frac{x - y}{x + y}.$$

By rewriting the equation in the form

$$\left(x\frac{dy}{dx} + y\right) + y\frac{dy}{dx} = x,$$

find and sketch the family of solutions.

Solve the same equation using the substitution $y = ux$.

7. *How to determine the properties of the solution to a differential equation without solving it.* Measurements on a yeast culture have shown that the rate of increase of the amount, or biomass, of yeast is related to the biomass itself by the equation

$$\frac{dN}{dt} = aN - bN^2,$$

where $N(t)$ is a measure of the biomass at time t, and $a, b > 0$. Without solving the equation, find in terms of a and b:

   (a) the value of N at which dN/dt is a maximum;

   (b) the values of N at which dN/dt is zero, and the corresponding values of d^2N/dt^2.

   Using all this information, sketch the graph of $N(t)$ against t.

   Now solve the equation analytically, giving your answer in terms of tanh.

8. Water flows into a cylindrical bucket of depth H and cross-sectional area A at a volume flow rate Q which is constant. There is a hole in the bottom of the bucket of cross-sectional area $a \ll A$. When the water level above the hole is h, the flow rate out of the hole is $a\sqrt{2gh}$, where g is the gravitational acceleration. Derive an equation for dh/dt. Find the equilibrium depth h_e of water, and show that it is stable.

9. In each of the following equation for $y(t)$, find the equilibrium points and classify their stability properties, by plotting $\dot{y}$ against y:

   (a) $\dot{y} = y(y-1)(y-2)$;

   (b) $\dot{y} = -2\arctan(y/(1+y^2))$;

   (c) $\dot{y} = y^3(e^y - 1)^2$.

10. Investigate the stability of the constant solutions ($u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = 4u_n(1-u_n).$$

In the case $0 \le u_0 \le 1$, use the substitution $u_0 = \sin^2\theta$ to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case $u_0 > 1$?

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