Examples Sheet 1

In Differential Equations, you will have to be familiar with both the Leibniz notation $\frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^nf}{dx^n}$ etc., and the Lagrange notation $f'(x), f''(x), f^{(n)}(x)$ etc. For partial differentiation, the Leibniz notation is $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x^2}$ etc., and the Lagrange notation is f_x, f_{xy}, f_{xx} etc.

For graphing functions, you may consider using online software such as geogebra or desmos to help you.

1. Using the Leibniz rule $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$ or otherwise, calculate
   (a) $\dfrac{d^n}{dx^n}(xe^{ax})$,
   (b) $\dfrac{d^n}{dx^n}(x^2e^{ax})$,
   (c) $\dfrac{d^{2n}}{dx^{2n}}(x^2 \sinh x)$.

   Finding the Taylor series of a function is like finding the decimal expansion of a real number. It gets more and more accurate as you increase the number of terms. Usually.

2. For some functions, there are shortcuts to finding the Taylor series.
   (a) *Using known functions.* Find the Taylor series for $f(x) = \frac{1}{1+x^2}$ by substituting $x \mapsto -x^2$ in the expansion for $\frac{1}{1-x}$ which you have found from the previous sheet. Write your answer in the form $\sum_{n=0}^{\infty}(\dots)$. Also write out the first four nonzero terms using the O notation. Sketch the function $\frac{1}{1+x^2}$ and also its Taylor series (*are they the same?*)
   (b) *Integration.* By integrating the Taylor series for $\frac{1}{1+x^2}$ found above from 0 to u, find the Taylor series for $\arctan u$. Write your answer in the form $\sum_{n=0}^{\infty}(\dots)$. Also write out the first four nonzero terms using the O notation.
   (c) *Binomial expansion.* Recall $(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$. Here n need not be an integer. An example is:

$$\begin{aligned}(1+x)^{-1/3} &= 1 - \frac{1}{3}x + \frac{(-\frac{1}{3})(-\frac{4}{3})}{(1)(2)}x^2 + \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{(1)(2)(3)}x^3 + O(x^4) \\ &= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + O(x^4).\end{aligned}$$

   Find the first four nonzero terms of the Taylor series for $f(x) = \sqrt{x^2+1}$ and the first three for $g(x) = \frac{1}{\sqrt[4]{1-ax}}$.
   (d) *Multiplication.* Find the first 3 nonzero terms of the Taylor series about $x = 0$ of $e^{ax}\sin(bx)$ ($a, b \neq 0$ constants) by quoting the expansion for e^{ax} and $\sin(bx)$ and then multiplying out, neglecting higher order terms.
   (e) *Reciprocal.* Find the first 3 nonzero terms of the Taylor series about $x = 0$ of $\sec x$ by writing it as the reciprocal of $\cos x$. Avoid any unnecessary calculations.
   (f) *Composition.* By writing $\ln(\cos x)$ as $\ln(1 - (\frac{x^2}{2!} - \frac{x^4}{4!} + O(x^6)))$ and quoting the expansion of $\ln(1-x)$, find the first two nonzero terms of the Taylor series of $\ln(\cos x)$.

3. *More Taylor series.* Using techniques/results from the previous question, or otherwise, find (using $\sum$ notation or the big O notation) the first n nonzero terms of the Maclaurin series of:

(a) e^{x^2}, $n = \infty$;
(b) $e^x \sin x$, $n = 4$;
(c) $\sqrt[3]{x^3 + a^3}$, $n = 3$;
(d) $\frac{\ln(1+x)}{1+x}$, $n = \infty$;
(e) $\arcsin x$, $n = 3$;
(f) $\ln(\frac{1+x}{1-x})$, $n = \infty$;

4. (a) *Taylor series about a general point a.* By using Taylor's formula

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

compute the Taylor series for $f(x) = e^{ax}$ about $x = 1$.

(b) Write down the Taylor series for $\ln(1 + x)$ about $x = 0$. Then show that

$$\lim_{k\to\infty} k \ln(1 + \frac{x}{k}) = x$$

and deduce that

$$\lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k = e^x.$$

5. *Partial Derivatives.* Let $f(x, y) = e^{-xy}$.

(a) Find f_{xy} and f_{yx} and check they are equal (note f_{xy} means first take $\frac{\partial}{\partial x}$ then take $\frac{\partial}{\partial y}$).

(b) By expressing f in terms of the polar coordinates r, θ, find f_r and f_θ.

(c) By using the chain rule

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta,$$

(and a similar version for f_θ), find f_r and f_θ. Check that the two methods give the same results. (Recall: $x = r\cos\theta, y = r\sin\theta$.)

(d) *Why it is important to specify which variables we are holding constant.* Compute $\left.\frac{\partial r}{\partial x}\right|_y$ and $\left.\frac{\partial r}{\partial x}\right|_\theta$ and check that they are not equal (recall: $r = \sqrt{x^2 + y^2}$). It is usually not possible to specify which variables are held constant when using the Lagrange notation, hence here we are using the Leibniz notation. From now on, if we do not specify which variable is held constant, we will just assume the obvious one. *This is an important point; I've had students ask me this over and over again in Lent term.*

6. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (*)$$

for $u(x, y)$ by making a change of variables as follows. Define new variables $\xi = x-y, \eta = x$, and evaluate the partial derivatives of x and y with respect to ξ and η. Writing $v(\xi, \eta) = u(x, y)$, use these derivatives and the chain rule to show that

$$\frac{\partial v}{\partial \eta} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y},$$

and that the equation $\frac{\partial^2 v}{\partial \eta^2} = 0$ is equivalent to $(*)$. Deduce that the most general solution of $(*)$ is $u(x,y) = f(x-y) + xg(x-y)$ where f and g are arbitrary functions that are suitably nice. Solve $(*)$ completely given the boundary conditions $u(0,y) = 0$ for all y and $u(x,1) = x^2$ for all x.

7. Let u be a function of x, y and consider the change of variables

$$x = e^{-s}\sin t, \quad y = e^{-s}\cos t.$$

   (a) Use the chain rule to express $\partial u/\partial s|_t$ and $\partial u/\partial t|_s$ in terms of $x, y, \partial u/\partial x|y$ and $\partial u/\partial y|x$.

   (b) Find, similarly, an expression for $\partial^2 u/\partial t^2$.

   (c) Hence transform the equation

$$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = 0$$

   into a partial differential equation involving s and t partial derivatives. *Note: product rule will need to be used when you encounter things like* $y\frac{\partial}{\partial x}(x\frac{\partial u}{\partial y})$.

Differentiating Under the Integral Sign. This technique can help us to find exact values of definite integrals without calculating the indefinite integral. The famous Richard Feynmann does this all the time when his friends couldn't work out an integral using usual techniques.

8. Sketch the function $f(x) = \frac{\sin \lambda x}{x}e^{-\alpha x}$ on $x \in [0,\infty)$ for (i) $\alpha = 0, \lambda > 0$; (ii) $\alpha > 0, \lambda > 0$. By differentiating I with respect to λ, show that

$$I(\lambda,\alpha) = \int_0^\infty \frac{\sin \lambda x}{x}e^{-\alpha x}\,dx = \arctan\frac{\lambda}{\alpha} + c(\alpha).$$

Here you may use differentiation under the integral sign without proof. Show that $c(\alpha)$ is zero by a choice of λ. Show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x}\,dx = \frac{\pi}{2}.$$

What is the value of the integral when $\lambda < 0$?

9. Let

$$f(x) = \left[\int_0^x e^{-t^2}\,dt\right]^2, \qquad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2}\,dt.$$

Show that $f'(x) + g'(x) = 0$. Deduce that $f(x) + g(x) = \frac{\pi}{4}$ and hence that

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

You may differentiate under the integral sign and use the Fundamental Theorem of Calculus without proof.

The following questions are optional.

10. *Even more Taylor series.* Find (using $\sum$ notation or the big O notation) the first n nonzero terms of the Maclaurin series of:

   (a) $\tan x$, $n = 3$;

   (b) $\int_0^x \frac{du}{1+u^4}$, $n = \infty$;

   (c) $e^{\sin x}$, $n = 4$;

   (d) $e^{\cos x}$, $n = 3$;

   (e) $e^{-\frac{1}{(x-a)^2}}$, where $a \in \mathbb{R}$ is a constant, $n = 3$. Use the convention $e^{-\infty} = 0$. What is special when $a = 0$?

11. Show that the change of variables $x = uv, y = 1/v$ reduces the equation

$$x^2 \frac{\partial^2 f}{\partial x^2} - 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + 2y \frac{\partial f}{\partial y} = 0 \qquad (*)$$

   to $\frac{\partial^2 F}{\partial v^2} = 0$, where $f(x,y) = F(u(x,y), v(x,y))$. Hence find the general solution of $(*)$.

12. (a) Let f be a real-valued function of real variables x and y having continuous second-order partial derivatives. If $x = r\cos\theta$ and $y = r\sin\theta$, show that if f is considered also as a function of r and θ, then, for $r \neq 0$,

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta}.$$

   Hint: express r and θ as functions of x and y and use the chain rule.

   (b) Find a similar expression for $\frac{\partial f}{\partial y}$.

   (c) Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

   (d) Suppose now that f satisfies Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ everywhere except possibly at the origin. Show that if f is symmetric under all rotations, regardless of angle, i.e. if $f(x,y) = g(x^2+y^2)$ for some differentiable function $g\colon \mathbb{R} \to \mathbb{R}$, then for some constant A, B, $f = A + B\log(x^2+y^2)$.

   (e) Suppose still that f satisfies Laplace's equation but now $f(r,\theta) = r^\alpha \Theta(\theta)$ for some real constant α. Show that α must be an integer and find the general form of $\Theta(\theta)$. *Note that the function must be single-valued, i.e. it takes only one value for each particular r and θ.*

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