Examples Sheet 0

The main content of this sheet is curve sketching and Taylor series. But first let's have something to start you off...

0. Practice writing all 24 Greek letters in alphabetical order, including upper case and lower case and learn their names. While you do not need to remember the order, you will need to know their names. It is important that you distinguish your ρ from your p, your ν from your v and your ω from your w as they may appear in the same equation. The hardest letter to write is ξ (ask the second years...) so practice writing it.

1. Some integration by parts... *For integration by parts multiple times, use the Tabular Method* (`http://people.math.harvard.edu/~knill/teaching/math1a_2012/handouts/39-parts.pdf`)

   (a) $\displaystyle\int \frac{1}{x} \ln x \, dx$

   (b) $\displaystyle\int e^{ax} \cos(bx) \, dx$

   (c) $\displaystyle\int_0^\pi (x^2-1)^2 \cos(2x) \, dx$

   (d) $\displaystyle\int_0^\pi x^5 \sin(3x) \, dx$

2. Prove that the sum to infinity of a geometric progression whose initial term is a and whose common ratio is r is $\frac{a}{1-r}$. For which range of values of r is this valid?

   Calculate the sum to infinity of $153 + \frac{153}{2} + \frac{153}{4} + \frac{153}{8} + \dots$. An arithmetic progression has first term 3.5 and common difference 4. Its sum is equal to that of the sum to infinity you have just calculated. Find the number of terms in the arithmetic progression and its final term.

3. Solve for θ in the interval $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$:

$$(\sin\theta - \cos\theta)(\sin^2\theta + 2\cos^2\theta) \leq 0.$$

4. Find the exact value of $\cot \frac{\pi}{12}$.

5. Solve $\cosh \log x = \frac{19}{8} + \sinh \log \frac{x}{4}$.

6. The equation $ax^2 + bx + c$ has roots α and β. Find $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ in terms of the coefficients a, b, c.

   The equation $ax^3 + bx^2 + cx + d$ has roots α, β and γ. Find $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$ in terms of the coefficients a, b, c, d.

7. *Curve sketching.* Sketch the curves on the same graph for each subquestion. Label the salient features of the graph.

   (a) $y = x^2 + 1$ and $y = \dfrac{1}{x^2+1}$

   (b) $y = x^2 - 1$ and $y = \dfrac{1}{x^2-1}$

   (c) $y = (1-x)^{-1/4}$ (do this in steps: first sketch $1-x$, then take its reciprocal...)

   (d) $y = \cos(x) - 1$ and $y = \cos(2x) + 1$ on $[0, 2\pi]$

   (e) $y = \cosh x$ and $y = \sinh x$

   (f) $y = (x-1)(x-2)$ and $y = |(x-1)(x-2)|$

   (g) $y = (x-1)(x-2)$ and $y = (|x|-1)(|x|-2)$

(h) the parametric equation $(x, y) = (\cos t, \sin t)$ and also the parametric equation $(x, y) = (\cosh t, \sinh t)$

(i) $y = e^x \sin x$

(j) $y = \sin\left(\frac{1}{x}\right)$

(k) $y = x^2 \sin\left(\frac{1}{x}\right)$

(l) $y = \frac{\sin x}{x}$

(m) $y = \arctan(\tan x)$ for $x \in (-2\pi, 2\pi)$

(n) $y = -\frac{1}{x^2}$ and $y = e^{-1/x^2}$

Taylor Series. *If you haven't learnt about Taylor Series, you should still try these questions.*

8. *Approximation of the cosine curve by polynomials.* Let $f(x) = \cos x$.
   (a) Find $f(0), f'(0), f''(0), f'''(0), f''''(0)$.
   (b) Find the quadratic $p_2(x) = a_0 + a_1x + a_2x^2$ that "best approximates" f at $x = 0$, by requiring $f(0) = p_2(0)$, $f'(0) = p_2'(0)$ and $f''(0) = p_2''(0)$. In other words, we are looking for the quadratic p_2 that has the same y-value, slope and convexity as cosine at the point $x = 0$. You should just get a system of simultaneous equations in the unknowns a_0, a_1, a_2.
   (c) Find the quartic $p_4(x) = \sum_{k=0}^4 b_k x^k$ that best approximates f at $x = 0$, by requiring $f^{(k)}(0) = p_4^{(k)}(0)$ for $k = 0, 1, 2, 3, 4$. Notation: $f^{(k)}$ is the kth derivative of f. So $f^{(2)} = f''$, $f^{(1)} = f'$, and by convention, $f^{(0)} = f$.
   (d) Sketch f, p_2 and p_4 on the same graph, using a computer to aid you if desired.

9. *Taylor's formula.* Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely-differentiable function, and $p_n(x) = \sum_{k=0}^n a_k x^k$ such that $p_n^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, ..., n$, i.e. the polynomial of degree $\leq n$ that best approximates f at $x = 0$. Show that
$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

10. *More Taylor series.* The Taylor series is what we get when we keep taking derivatives $k = 0, 1, 2, 3, \ldots$ forever. We can find the Taylor series of some elementary functions by computing $f^{(k)}(0)$. For example, if $f(x) = \cos x$, then $f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{(4)}(0) = 1, \ldots$ so by the formula above,
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$
    By computing $f^{(k)}(0)$ for all $k \geq 0$, find the Taylor series for
   (a) $f(x) = \sin x$;
   (b) $f(x) = e^x$;
   (c) $f(x) = \cosh x$;
   (d) $f(x) = \frac{1}{1-x}$ [you should get the geometric series];
   (e) $f(x) = \ln(1 + x)$.

   For each function above, graph f together with the nth order term p_n for a few values of n using a computer (if you type `sum` in desmos you can use summation notation). What happens when n is large?

11. Let J_n be the indefinite integral

$$J_n = \int \frac{x^{-n}\,dx}{(ax^2+2bx+c)^{1/2}}.$$

By integrating $\int x^{-n-1}(ax^2+2bx+c)^{1/2}\,dx$ by parts, show that for $n \neq 0$,

$$ncJ_{n+1} + (2n-1)bJ_n + (n-1)aJ_{n-1} = -x^{-n}(ax^2+2bx+c)^{1/2}.$$

Hence evaluate

$$\int_1^2 \frac{dx}{x^{5/2}(x+2)^{1/2}}.$$

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